APPLIED CALCULUS

NEWTON (1642–1727)

"QUI GENUS HUMANUM INGENIO SUPERAVIT"

From the painting by Vanderbank (National Portrait Gallery)

APPLIED CALCULUS

An Introductory Textbook

BY

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PREFACE

This book is intended to provide an introductory course in the Calculus for the use of students of natural and applied science whose knowledge of mathematics is slight. All the mathematics that the student is assumed to know is algebra up to quadratic equations; elementary trigonometry up to the formulæ of sines, cosines, and tangents of compound angles; the elements of geometry; and the method of graphs.

Infinite series are essentially difficult and unconvincing unless treated rigorously—as the old conundrum of Achilles and the tortoise shows—and there is no need to use them in the elementary parts of the subject. They have therefore been avoided altogether.

Definite problems, dealing with actual things, precede the analytical treatment, which I have tried to make simple and convincing; and I hope any reader who pursues the subject further in the standard works will find that he has only to extend and qualify the proofs, not to unlearn them.

I have introduced and used limits in the first chapter before *defining* them, for the same reason that I should show a child a herring and tell him about its habits of life before describing it to him as one of two distinct but closely-allied species of malacopterygian fishes of the genus Clupea.

The pictures of celebrated mathematicians and scientists are intended to arouse some human interest in mathematical science and the history of its progress. Some of the founders of the science lived more than ordinarily interesting lives, and if the mathematician ignores the human side of things, he can hardly expect humanity not to ignore him.

Perhaps the title of the book needs a word of explanation. In applied mechanics it is usual to discuss the theoretical principles of mechanics as well as their applications. This line has been followed here, the treatment of practical problems being preceded by a fairly full discussion of the necessary theory.

The examples appended to the several chapters have been specially chosen to illustrate particular points, and should all be worked by the student. It has not been thought necessary to supply long sets of examples, such sets being rarely worked through by the student. To help the private student, whose needs I have had specially in mind, solutions of all the difficult problems are given, and answers to all exercises and problems. Many of the exercises, especially those in the "Miscellaneous Exercises", are taken by permission from recent University and Army Examination Papers. I gratefully acknowledge my indebtedness to the authorities of the Universities of Cambridge, London, and Glasgow, and to the Comptroller of H.M. Stationery Office, for permissions granted.

I wish also to thank Mr. J. Dougall, M.A., D.Sc., F.R.S.E., for the great help he has given me in passing the book through the press. He has read all the proofs, and has made many valuable suggestions. Mr. D. J. Richards, M.A., also kindly read the MS. in the earlier stages, and I am indebted to him for numerous criticisms.

The books to which I am most indebted are G. H. Hardy's *Pure Mathematics*, on critical points, and the treatises of Lamb and Gibson. I have also found Dr. A. N. Whitehead's book, *An Introduction to Mathematics*, most suggestive. The data used and methods described in the chapter on physical chemistry are based on the classic work of J. H. Van 't Hoff, *Lectures on Theoretical and Physical Chemistry*, and the treatise of W. C. M'C. Lewis, *A System of Physical Chemistry* (2 vols.). I have also consulted R. A. Lehfeldt's *A Textbook of Physical Chemistry*.

F. F. P. B.

GLASGOW, *July*, 1921.

The publishers gratefully acknowledge the kindness of Miss Anna Guldberg and Professor Torup of the University of Christiania in sending them a biography of C. M. Guldberg (p. 412) and the photograph from which the half-tone facing p. 306 was reproduced.

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APPLIED CALCULUS

Introduction

"What a time of it had we, were all men's life and trade still, in all parts of it, a problem, a hypothetic seeking, to be settled by painful Logics and Baconian Inductions! The Clerk in Eastcheap cannot spend the day in verifying his Ready Reckoner; he must take it as verified, true and indisputable; or his Book-keeping by Double Entry will stand still. 'Where is your Posted Ledger?' asks the Master at night. 'Sir,' answers the other, 'I was verifying my Ready Reckoner and find some errors. The Ledger is——!' Fancy such a thing!

"True, all turns in your Ready Reckoner being moderately correct, being *not* insupportably incorrect."—CARLYLE.

In the year 1612, there was an unusually heavy vintage in Germany, and it appeared that there would not be enough barrels to hold all the wine. The amount of wine which would go into a barrel of given shape and dimensions could not be calculated, so the astronomer Kepler at once took the problem in hand. For many years, he had been working at problems which required the finding of areas contained by closed curves, and he had invented special methods of his own for solving these problems. These methods he applied to the wine tun, and wrote a book called *Nova Stereometria Doliorum* (a new method of measuring wine casks), published in 1615.

This book is the earliest work on the subject now called "The Integral Calculus".

Before explaining the method Kepler used, it will clear the ground if we deal with a few points of elementary geometry.

Position of a Point in a Plane.

The position of a point in a plane may be specified in the following way.

Suppose the plane in question is the plane of this paper.

Rule two straight lines, X′OX, Y′OY, of unlimited length, perpendicular to each other, and intersecting in the point O, called *the Origin* (fig. 1).

These lines divide the plane into four quadrants. If we were drawing a map, we should call these quadrants,

the N.E. quadrant,
N.W. ,,
S.E. ,,
S.W. ,,

Take a point P (fig. 1) anywhere in the N.E. quadrant.

Draw PM perpendicular to X′OX, meeting X′OX in M.

If we measure the length of OM and of MP we shall know the exact position of P in the N.E. quadrant with reference to X′OX and Y′OY.

Thus, suppose the length of OM to be 1 in. and that of MP 2 in., then P is 1 in. from Y′OY, and 2 in. from X′OX.

It is usual to denote OM by x and MP by y. The two lengths denoted by x and y are called the *co-ordinates* of P. When it is necessary to distinguish between them, x is called the *abscissa* and y the *ordinate*.

The point P is referred to as the point (x, y). In the actual case taken above

$$x = 1 \text{ and } y = 2 \text{ and P is the point } (1, 2).$$

A common unit of length is understood in all measurements—the inch in the above case. The figure is half full size.

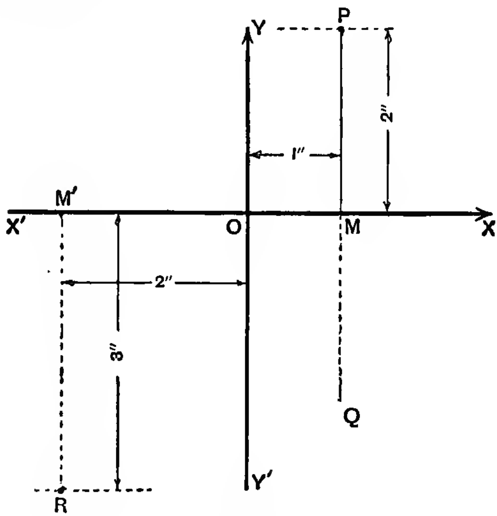


Fig. 1

The base lines X′OX and Y′OY are called the *axes*; X′OX, the *x axis*, and Y′OY, the *y axis*. All measurements are made from these lines, in directions perpendicular to them.

Convention of Signs.

Every number is associated with a sign, implied or stated. This sign is either plus or minus. In specifying the point P (1, 2), the positive sign is implied, i.e. P is the point (+ 1, + 2).

If a measurement is made in one direction when the number expressing it is positive, it must be made in the opposite direction when the number expressing it is negative.

This rule means that we regard *length* as a magnitude to which a plus or minus sign may be given. The directions usually taken as *positive* are shown by the arrow-heads at X and Y (fig. 1). Thus, a line drawn in the direction from O to X, or from O to Y, is positive, while a line drawn in the direction from O to X′, or from O to Y′, is negative. This rule at once enables us to specify points in the remaining three quadrants.

The point specified by (1, − 2), for instance, is *plotted*, i.e. drawn, by measuring off OM, equal to 1 in., and MQ, equal to 2 in., MQ being measured from X′OX in the direction opposite to OY.

Likewise, the point (− 2, − 3) is the point R (see fig. 1) where

OM′ is 2 in. long,
M′R is 3 in. long.

The point (− 2, 3) lies in the X′OY (the N.W.) quadrant.

It does not matter which quadrant we take as corresponding to (+ x, + y) points—it is usual to take the N.E. one—but it is essential to adhere to the sign convention.

The rule may be remembered by the mnemonic:

Plus, to the right; minus, to the left.
Positive, height; negative, depth.

The signs of x and y in the different quadrants are then

$$(x+, y+), \text{ N.E.}$$
$$(x-, y+), \text{ N.W.}$$
$$(x-, y-), \text{ S.W.}$$
$$(x+, y-), \text{ S.E.}$$

A Graph.

Consider the equation

$$\frac{x^2}{9} + \frac{y^2}{4} = 1 \quad\ldots\ldots\ldots\ldots(1)$$

This equation gives

$$\frac{y^2}{4} = 1 - \frac{x^2}{9}.$$

$$\therefore y^2 = 4\left(1 - \frac{x^2}{9}\right)$$

$$y = \pm 2\sqrt{1 - \frac{x^2}{9}}. \quad\ldots\ldots\ldots\ldots(2)$$

This formula enables us to calculate y when x is given, but there is evidently a limit to the admissible values of x. No ordinary number can be equal to the square root of a negative number; for if p were such a number equal to the square root of, say, -4,

$$p = \sqrt{-4},$$
$$\text{then } p^2 = -4.$$

But the square of any number, be it positive or negative, is a *positive* number.

Hence, if y is to be an ordinary number, $\left(1 - \frac{x^2}{9}\right)$ must be positive.

$\left(1 - \frac{x^2}{9}\right)$ will be positive so long as x^2 does not exceed 9, i.e. so long as x is not greater than 3, numerically.

When x is -3, the value of $\frac{x^2}{9}$ is 1, and when x is $+3$, the value of $\frac{x^2}{9}$ is 1, and for all values of x between -3 and $+3$, the value of $\frac{x^2}{9}$ is less than 1.

The permissible values of x lie, therefore, between

$$x = -3 \text{ and } x = +3,$$

and on referring to equation (1) it is evident that when x has either of these values, y must be zero. We will now give x a series of values between -3 and $+3$, and calculate y. A few corresponding values are given below.

x	-3	-2	-1	0	$+1$	$+2$	$+3$
y	0	$\pm\frac{2}{3}\sqrt{5}$	$\pm\frac{2}{3}\sqrt{8}$	± 2	$\pm\frac{2}{3}\sqrt{8}$	$\pm\frac{2}{3}\sqrt{5}$	0

These numbers are sets of values for x and y which satisfy the equation between x and y

$$\frac{x^2}{9} + \frac{y^2}{4} = 1. \quad \text{.....................(3)}$$

It will be seen that, corresponding to a given value of x (say, $+2$), there are *two* possible values of y, namely, $(+\frac{2}{3}\sqrt{5})$ and $(-\frac{2}{3}\sqrt{5})$.

Now, if the numbers are all supposed to stand for lengths, we can treat these values of x and y $(-1, -\frac{2}{3}\sqrt{8}$, e.g.) as co-ordinates of a set of points in a plane and plot the corresponding points. We shall thus get a picture, so to speak, of pairs of numbers which satisfy the given equation.

It is convenient to use squared paper for plotting. The points are plotted in fig. 2.

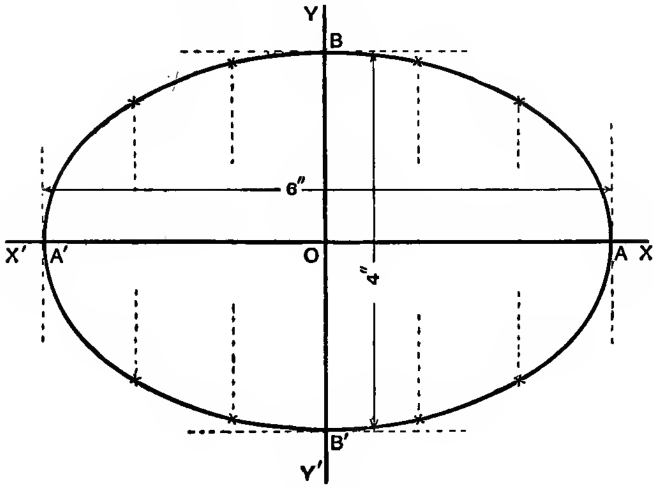


Fig. 2

A fair curve, suggesting an ellipse, can be drawn through these points.

The "graph" of a mathematical equation is a line in a plane so drawn that the co-ordinates of every point in it satisfy the given equation.

The line may be either a straight line or a curved one.

Thus the curve roughly outlined in fig. 2 is the *graph* of $\frac{x^2}{9} + \frac{y^2}{4} = 1$.

We can show that the graph of the equation

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

is an ellipse, but before we do this we must say what we mean by an "ellipse".

Definition and Geometrical Construction of an Ellipse.

Suppose a point moves in a plane in such a way that the sum of the distances of the point from two fixed points in the plane is constant. The moving point traces out a path, and this path is called an *ellipse*.

The curve may be drawn in the following way.

Suppose P and Q are the given fixed points (fig. 3).

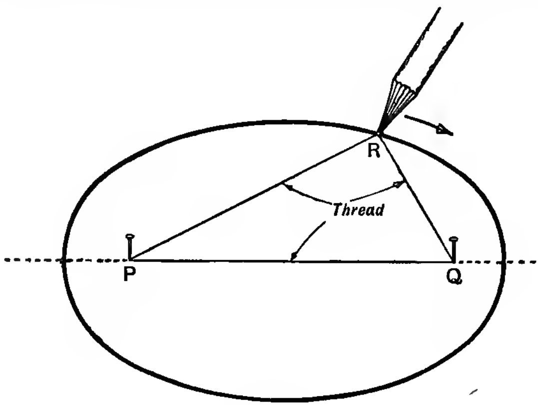


Fig. 3

Fix pins at P and Q, and tie a loop of thread round the pins so that, when stretched out tight, its total length (PQ + QR + RP) is equal to the given sum of the distances of the moving point from P and Q, plus PQ. Since PQ is constant, PR + RQ must be constant, wherever R may be. Place a pencil point at R, as shown in fig. 3, and move it round. It will trace out a curve which possesses the property that PR + RQ is constant wherever R may be. The curve is therefore, by definition, an ellipse.

We will now use this property to find the equation which must be satisfied by the co-ordinates of the moving point.

The Equation of an Ellipse.

Join the given fixed points P and Q, and produce the line indefinitely, X'X (fig. 4).

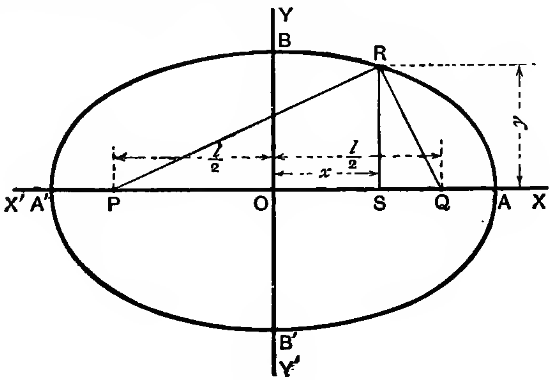


Fig. 4

Bisect PQ at O and through O draw Y'Y perpendicular to PQ of unlimited length.

Take O as the origin, and X'X as axis of x and Y'Y as axis of y.

Let R be a point on the ellipse.

The co-ordinates of R are

$$x = OS$$
$$y = SR.$$

Now PR + RQ must be constant (the property of the ellipse).

Let the constant be m. Then

$$\text{PR} + \text{RQ} = m. \quad \dots\dots\dots\dots\dots\dots(4)$$

The length PQ is also given.

$$\text{Let PQ} = l$$

$$\therefore \text{OQ} = \frac{l}{2}. \quad \dots\dots\dots\dots\dots\dots(5)$$

$$\text{Also PR} = \sqrt{\overline{\text{PS}}^2 + \overline{\text{SR}}^2}$$

$$= \sqrt{\overline{\text{PO} + \text{OS}}^2 + \overline{\text{SR}}^2}$$

$$= \sqrt{\left(\frac{l}{2} + x\right)^2 + y^2}$$

$$\text{and RQ} = \sqrt{\overline{\text{SQ}}^2 + \overline{\text{SR}}^2}$$

$$= \sqrt{\left(\frac{l}{2} - x\right)^2 + y^2}.$$

$$\therefore \sqrt{\overline{\frac{l}{2} + x}^2 + y^2} + \sqrt{\overline{\frac{l}{2} - x}^2 + y^2} = m \text{ by (4)}.$$

To simplify this equation, we have, on squaring,

$$\overline{\frac{l}{2} + x}^2 + y^2 + \overline{\frac{l}{2} - x}^2 + y^2$$

$$+ 2\sqrt{\left(\overline{\frac{l}{2} + x}^2 + y^2\right)\left(\overline{\frac{l}{2} - x}^2 + y^2\right)} = m^2.$$

$$\therefore x^2 + y^2 + \frac{l^2}{4} - \frac{m^2}{2}$$

$$= -\sqrt{\left(\overline{\frac{l}{2} + x}^2 + y^2\right)\left(\overline{\frac{l}{2} - x}^2 + y^2\right)}.$$

$$\therefore \left[x^2 + y^2 + \frac{l^2}{4} - \frac{m^2}{2}\right]^2$$

$$= \left(\overline{\frac{l}{2} + x}^2 + y^2\right)\left(\overline{\frac{l}{2} - x}^2 + y^2\right).$$

Simplifying this equation, we get

$$\frac{m^2}{4}(m^2 - l^2) - x^2(m^2 - l^2) - y^2m^2 = 0. \quad \ldots\ldots(6)$$

When R is at B, PB = BQ and 2 PB = m; hence if

$$OB = b,$$

$$2\sqrt{\frac{l^2}{4} + b^2} = m.$$

$$\therefore\ b^2 = \frac{m^2 - l^2}{4}. \quad \ldots\ldots\ldots\ldots(7)$$

Similarly, when R is at A,

$$PA + QA = m = PQ + 2\,QA;$$

and, if OA = a, QA = $a - \frac{l}{2}$.

$$\therefore\ l + 2\left(a - \frac{l}{2}\right) = m.$$

$$\therefore\ 2a = m.$$

$$\therefore\ a = \frac{m}{2}. \quad \ldots\ldots\ldots\ldots\ldots(8)$$

We can therefore rewrite (6) as

$$4a^2b^2 = 4b^2x^2 + 4a^2y^2.$$

$$\therefore\ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

This equation is therefore that of an ellipse. a and b are constants which settle the size and shape of the ellipse. O, the origin in fig. 4, is called the *centre*; OA, which is equal to a, the semi-major axis; and OB, which is equal to b, the semi-minor axis of the ellipse.

The graph of

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

is therefore an ellipse whose centre is at O, and whose major axis lies along the x axis and minor axis along the y axis. The semi-major axis is 3, and the semi-

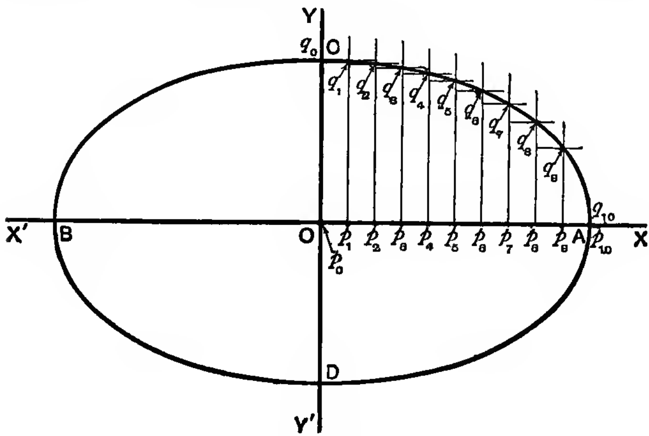


Fig. 5

minor axis 2 units long. In fig. 2 the inch is the unit of length.

EXAMPLES FOR PRACTICE

Draw the graphs of the following mathematical equations:—

(i) $y = 2x + 3$.
(ii) $x^2 + y^2 = 4$.
(iii) $y^2 = x$.

The Area of an Ellipse—Kepler's Method.

Suppose we have to find the area enclosed by the ellipse shown in fig. 5.

It is evident from the symmetry of the figure that we need only find the area of the portion lying in the positive quadrant, for if A stands for this area then 4A stands for the required area.

Divide OA into a certain number, say 10, of equal parts by the points of division p_1, p_2, p_3, &c. p_0 coincides with O and p_{10} with A. Through these points of division draw straight lines parallel to Y′OY, meeting the ellipse in points q_0, q_1, q_2, &c. q_0 coincides with C, and q_{10} with p_{10} and A.

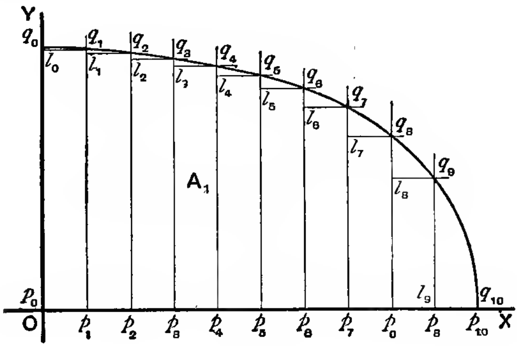


Fig. 6

Through each of these points q_0, q_1, q_2, &c., draw straight lines parallel to X′OX to meet the neighbouring ordinates in v and l, i.e. the line parallel to X′OX through q_5 is drawn to intersect the line p_4q_4 in l_4 and the line p_6q_6 in v_6.

We can thus construct two *rectilinear figures*, shown in figs. 6 and 7, one of which lies wholly within the given figure OAC, while the given figure lies wholly within the other.

There is no difficulty in calculating the areas of these rectilinear figures as each is composed of a set of rectangles. The length and the breadth of each of these rectangles are known.

The area of the figure (p_0, l_0, q_1, l_1, q_2, l_2, . . . l_8, q_9, p_9, p_0) (fig. 6) is evidently less than the required area, while the area of the figure (p_0, q_0, v_1, q_1, v_2, q_2, . . . v_{10}, p_{10}, p_0) (fig. 7) is greater than the required area.

Let A_1 stand for the area of the rectilinear figure

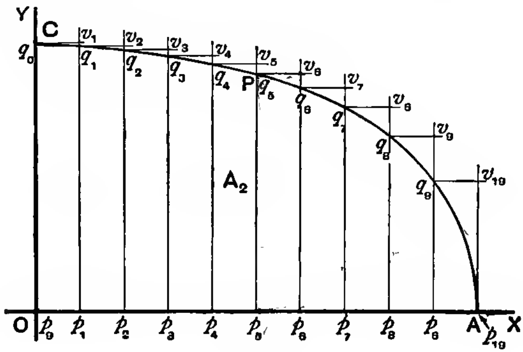


Fig. 7

shown in fig. 6, and A_2 for that shown in fig. 7. Then, if A stands for the true area required, A is greater than A_1 and less than A_2, i.e. A lies between A_1 and A_2.

This fact may be written

$$A_1 < A < A_2$$

where < is the sign for "is less than".

The error in taking A_2 as the required area does not exceed

$$\left\{\left(\frac{A_2 - A_1}{A_1}\right) \times 100\right\} \text{ per cent.}$$

Now comes the important point of the method. *By making the steps* p_0p_1, p_1p_2, p_2p_3, &c., *small enough, we can make* A_1 *and* A_2 *differ by as little as we please*, i.e. we can make the error as small as we please.

This statement will be considered more fully in the sequel. Meantime the student may easily verify that in figs. 6 and 7 the difference $A_2 - A_1$ is the sum of 10 rectangles, such as $l_4q_5v_5q_4$, which have a common breadth $a/10$.* The sum of the areas of these 10 rectangles is the product of their common breadth by the sum of their heights. Thus $A_2 - A_1 = \frac{1}{10}ab$. By taking more divisions, the factor corresponding to $\frac{1}{10}$ can be made as small as we please.

We will now consider a convenient notation for expressing the areas of the rectilinear figures shown in figs. 6 and 7. Consider any point P (fig. 7) lying on the curve AC.

The area of the strip $q_5p_5p_6v_6$ is $(p_5q_5) \times (p_5p_6)$. If p_5q_5 and p_5p_6 are measured in inches, the product is the area in square inches of the rectangle in question.

Let the co-ordinates of P be (x, y).

When the breadth of the strip (p_5p_6) is small enough for our purpose, we will call it the "difference in x" and write it δx.

Compare $\sqrt{}$, which stands for "square root of". Neither δ nor $\sqrt{}$ stands for a number, and so δx no more means $\delta \times x$ than $\sqrt{x}$ means $\sqrt{} \times x$. δx is simply the symbol for the width of the strips measured in the x direction, it being understood that δx is small enough to ensure that the difference in A_2 and A_1 is small enough for our purpose.

y stands for the length p_5q_5, and δx for the breadth p_5p_6, $\therefore$ area of strip $= y \times \delta x = y\delta x$.

Now, by construction, the width of each strip is the

* $a = Op_{10}$ = semi-major axis,
$b = Oq_0$ = semi-minor axis.

same, therefore δx stands for the width of each and every strip.

If y_1 is the ordinate of q_1,
y_2 ,, ,, q_2, and so on,

the whole area A_2 is evidently the sum

$$(y_0\delta x + y_1\delta x + y_2\delta x + \ldots y_9\delta x).$$

The co-ordinates of q_0 are $(0, y_0)$.
,, ,, q_1 ,, $(\delta x, y_1)$.
,, ,, q_2 ,, $(2\delta x, y_2)$.
. . . .
,, ,, q_9 ,, $(9\delta x, y_9)$.

The sum required is therefore obtained by (1) making P progress in steps, δx, from $x = 0$ to $x = a$; (2) taking the corresponding value of the ordinate to the curve at the beginning of each step in x, beginning with $x = 0$; and (3) multiplying δx by each of these ordinates and summing up the products so obtained.

This sum is written briefly

$$\sum_0^a y\,\delta x$$

where Σ stands for the words "the sum of terms of type", and 0 and a indicate the bounds of x between which the sum is to be taken.

We have seen that when the steps are made small enough, the error in taking A_2 as the required area can be made practically to disappear. In this case, the area A_2 *is indistinguishable from the area required*, A.

When δx is so reduced, a special symbol—an old-fashioned "s" ($\int$)—is used instead of the usual Σ (sigma) and d instead of δ, and we write

$$A = \int_0^a y\,dx.$$

The mathematical expression

$$\int_0^a y\,dx$$

stands for the area OAC (fig. 7) when y and x are connected by an equation of which the curve AC is the graph. One method of calculating the value of $\int_0^a y\,dx$, approximately, is to divide the area up into strips as shown in figs. 6 and 7; sum the areas of these strips; and repeat the process, using smaller and smaller strips, until the inscribed (A_1, fig. 6) and

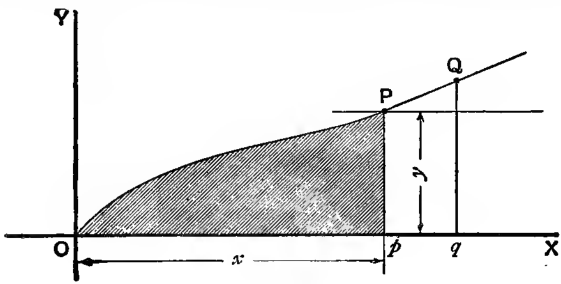


Fig. 8

escribed (A_2, fig. 7) rectilinear figures differ by a sufficiently small amount. This method is an easy and useful, though tedious, way of arriving at the required area, *within known limits of error*. The last words are important. An approximate result may be all that is required, if we know its limits of error. The problem of finding the value of $\int_0^a y\,dx$ *accurately* is the task of the integral calculus.

On the other hand, the differential calculus is concerned with problems like the following one.

Consider the curve OP (fig. 8). Let A stand for the area OpP.

Suppose we divide Op into a large number of equal parts. These parts need not be inches. They can be millionths of inches theoretically, and in actual drawing it is easy to make them $\frac{1}{10}$ in. Let Op increase by one part, i.e. by pq. The area added is PpqQ.

If we write ΔA for the increase in area,
and Δx ,, ,, x,
since $pq = \Delta x$,
we get $\Delta A = p\mathrm{P} \times \Delta x$, nearly.

$$\text{i.e. } \frac{\Delta A}{\Delta x} = p\mathrm{P} = y, \text{ nearly.} \quad \ldots\ldots(10)$$

∴ *the ordinate at* P *is nearly equal to the rate at which the area increases with x in the neighbourhood of* P, and by making pq progressively smaller and smaller, $\left(\frac{\Delta A}{\Delta x}\right)$ usually becomes progressively nearer and nearer to y in value.

We have, then, to examine in detail the approximate equation

$$\frac{\Delta A}{\Delta x} = y, \text{ nearly.} \quad \ldots\ldots\ldots\ldots\ldots(11)$$

This problem is the rôle of the Differential Calculus.

CHAPTER I

General Principles

"Observe always that everything is the result of a change, and get used to thinking that there is nothing Nature loves so well as to change existing forms and to make new ones like them."—MARCUS AURELIUS.

Concrete Number—Quantity.

A child sees some apples on a plate, and begins to wonder whether there are enough to go round. The question, "*How many* apples are there?" arises at once, because the answer is of much interest to him. His interest does not arise from any theoretical importance he attaches to the idea of quantity in itself—it is purely practical. He soon learns to count, and finds that if there are 4 apples and 4 people, he may get one, but if there are 2 apples and 4 people, the position is not so satisfactory. He thus comes to associate *numbers* with groups of things. When he does this, he is beginning to apply mathematics to practical problems. In all practical applications of mathematics, whether by the child or the skilled mathematician, the numbers used refer to *definite* quantities of a thing of some kind. They are concrete numbers or numerical quantities. The "thing" need not be a material thing in the sense that it can be weighed, though it may be. The quantity is always measured in terms of a "unit", and the number associated with the unit is the measure of the quantity in terms of the unit employed.

Suppose we are considering the length of the straight line AB (fig. 1).

If we use centimetres, the length will be represented by 6—if we use inches, by 2·36, and so on. The length can therefore be represented by *any* number, depending on the "unit" length we use.

To the purely arithmetical question, "What results when 3 is multiplied by 2?" the answer is 6; but to the question, "What is the area of a rectangular field whose sides are 3 and 2?" no answer can be given until we know the "unit", in terms

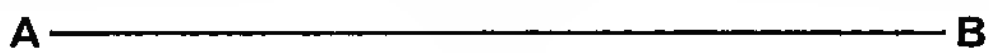


Fig. 1

of which the sides are measured. If they are measured in furlongs, the answer is 3 fur. × 2 fur. or 6 sq. fur.

It is not usually necessary to write after each figure the unit implied in it, but the unit must, nevertheless, be carefully kept in mind.

In physical science, the units employed are carefully chosen so as to keep calculations as simple as possible. For instance, although the area of a rectangle is proportional to the product of its length and its breadth, whatever units of length and of area are used, the area of a rectangle 3 in. long and 2 in. wide is represented by 6 only if the unit of area used is the square inch. If the inch is taken as the unit of length, the square inch is the natural unit of area.

Abstract Number.

In arithmetic, numbers are sometimes considered without any reference to particular objects.

$$3 \times 2 = 6$$

expresses a fact concerning the numbers 3, 2, and 6 considered as mere numbers. It is true no matter what things the numbers refer to.

We must distinguish between:

1. The thing or quality we are measuring;
2. The amount of that thing, i.e. the quantity of it;
3. The *standard* amount of the thing—the unit, i.e. the *arbitrary* amount we denote by the number 1; and
4. The number which measures the quantity of the thing in terms of the chosen unit.

The study of numbers themselves is only of indirect interest, since the numbers have only artificial meanings so long as they are divorced from a unit. Not till the unit is introduced does the number become

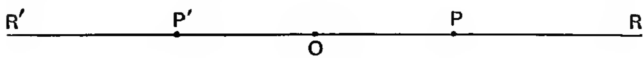


Fig. 2

a definite quantity standing for the amount of a thing. It is just as easy to be specific as to what the unit is, and, unless the contrary is stated, we shall suppose the unit is a given length, 1 in., say, of a straight line. Any number, therefore, represents a straight line, i.e. a quantity of length. For instance, if we measure from a starting-point at O, to the right (fig. 2), OP is the quantity represented by 3 if OP is 3 in. long.

Similarly, OP′ is the quantity represented by -3 if it is 3 in. long and measured in the opposite direction to that of OP.

Every point, R, R′, &c., in the straight line R′OR, when extended indefinitely in each direction, is the end of a straight line, the beginning of which is at O, and this straight line is a quantity of length corresponding to some number of units of length. Thus:

if OR is x in. long,
OR corresponds to the number x.

The straight line R′OR is therefore, as it were, a picture of all the numbers, as any point in it corresponds to some number.

The reader will recall that the whole theory of dimensions in physics is based on the essential difference between "quantity", or the amount of a thing, and the mere number that measures this amount in terms of the chosen unit. The mere number has no dimensions; the quantity has physical dimensions which are the same as those of the unit chosen.*

Change.

"All the world 's a stage,
And all the men and women, merely players;"

said Jaques. Some people play dumb parts that they may watch the heroes and the heroines the better—but play some part they must in the ever-changing scenes of the pageant of life and of nature.

Change is everywhere—stagnation is anathema. Change seems to be the stimulus that awakens consciousness. The child notices the changing colours on the wall, the moving leaves on the trees, and remembers them. The idea of change or variation is a fundamental one, and is derived from our experience

* For a full explanation of this theory, see the chapter on Dimensions in any standard work on physics, e.g. Deschanel, *Electricity and Magnetism*, pp. 346–50.

of things from earliest infancy. If a thing which is being measured changes, the quantity of it must change too.

The idea of variable quantity therefore arises. It arises directly from our fundamental idea of change.

On the other hand, there are certain things which we are led to believe do not change. The quantity of anything which does not change must remain constant, and will be represented in any given system of units by a definite unchanging number such as 2 or 3.

Variable Quantities—Functions.

In elementary mechanics, it is shown that, if a stone falls freely to the earth under the influence of gravity, the distance through which it falls in a given interval is given by

$$s = \tfrac{1}{2}gt^2, \quad \text{......................(1)}$$

where s stands for the vertical distance fallen through, in feet;

g stands for the acceleration due to gravity, in feet per second per second; and

t stands for the period of flight, in seconds.

The acceleration due to gravity does not change, appreciably; hence it will be measured by a constant number, which is 32 nearly, in the unit stated. So (1) becomes

$$\begin{aligned} s &= \tfrac{32}{2}t^2 \\ &= 16t^2. \quad \text{......................(2)} \end{aligned}$$

A stone may fall for a longer or a shorter period, and may, in consequence, travel a longer or a shorter journey. The values of s and t in (2) are therefore not unalterable numbers.

For instance, if $t = 1$, $s = 16$, i.e. the stone falls through 16 ft. in 1 sec.

If t is put equal to 2, we find $s = 16 \times 2^2 = 64$, i.e. the stone falls through 64 ft. in 2 sec.

So we can go on putting any positive number for t, and calculating the corresponding value of s. We can plot a graph of the results.

Quantities such as those denoted by s and t in (1) are termed *variable quantities*, while quantities, such as the acceleration due to gravity, which are essentially invariable, are called *constant quantities*.

It is usual to use the later letters of the alphabet, x, y, z, to stand for variable quantities, and the initial ones, a, b, c, &c., for constant quantities.

When two quantities are related to each other, so that changes in the value of one are accompanied by changes in the value of the other, each is said *to be a function of the other*.

The value of s, in the example, depends on the value given to t, while if s be given, the corresponding value of t depends on the particular value given to s. In fact the graph shows the correspondence which exists between the values of s and t. s is therefore a function of t, while t is a function of s.

This functional dependence is written

$$s = f(t)$$
$$\text{or } t = \psi(s),$$

where f, ψ stand for "a function of".

In this notation, $f(a)$ stands for the value of the function $f(x)$ when a is put for x; thus if $f(t)$ is $16t^2$, $f(a)$ is $16a^2$.

To take another example, suppose

$$y = 3x^2 + 2x + 2.$$

As y has different values, depending on the particular value given to x, we speak of y as being a function of x, and we symbolize the relationship thus:

$$y = \phi(x),$$

and on putting a for x,

$$\phi(a) = 3a^2 + 2a + 2;$$

or, on putting 2 for x,

$$\phi(2) = (3 \times 2^2) + (2 \times 2) + 2 = 18.$$

Different functional symbols, f, ψ, ϕ, &c., are used to distinguish functions *of different form*; thus:

$$\text{if } s = 16t^2,$$

$$t^2 = \frac{s}{16},$$

$$\text{and } t = +\frac{\sqrt{s}}{4},$$

we can write

$$s = f(t) = 16t^2 \text{ regarding } s \text{ as a function of } t,$$

$$\text{or } t = \psi(s) = \frac{\sqrt{s}}{4} \quad \text{,,} \quad t \quad \text{,,} \quad s.$$

Here f and ψ stand for functions of different form; if a be any variable, $f(a)$ is $16a^2$, while

$$\psi(a) \text{ is } \frac{\sqrt{a}}{4}.$$

EXAMPLES

1. If $\psi(x) = x^3$, what is the value of $\psi(2)$?
2. If $\phi(x) = x^3 + 3$, what is the value of $\phi(6)$?
3. If $\chi(x) = x^4 + 4a$, what is the value of $\chi(1)$?
4. If $f(x) = x^3 + 3x^2 + 3x + 1$, what is the value of $f(0)$?

Approximation.

A measurement is never *quite* accurate unless by accident; it is usually sufficient if it is *approximately* correct. Sometimes, indeed, an approximate measurement is all that is possible; for example, it would be difficult even to define *exactly* what is meant by the length of a given iron rod. How accurate the measurement should be depends on the purpose for which it is required. For example, suppose a pair of scales is made to turn when the weight on one side is $\frac{1}{100}$ oz. greater than the weight on the other. If an ounce of tea is weighed on such a pair of scales, the quantity of tea should be correct to within $\frac{1}{100}$ oz., i.e. to within 1 part in 100. In the same way, 1 lb. could be weighed with an error not exceeding $\frac{1}{100}$ oz. in 16 oz., i.e. to within 1 part in 1600. On the other hand, if such a pair of scales were used to weigh $\frac{1}{50}$ oz., the weight could be relied upon only to within 1 part in 2, and could make no pretence to accuracy. While an inaccuracy of $\frac{1}{100}$ oz. is negligible for *relatively* large amounts, it is of serious consequence when the amounts are relatively small. It is often difficult, and sometimes impossible, to *calculate the absolutely accurate value of a number*, though an approximate value may be easily obtained. People have been trying to find the accurate ratio of the circumference to the diameter of a circle for about four thousand years. It was not till quite recently that mathematicians succeeded in showing that this could not be done. For many purposes, the result (3·1604) obtained 3600 years ago was accurate enough, and a result less accurate but within about 5 per cent of the value accepted to-day (3·1416) is implied in the Old Testament (1 Kings, vii, 23):

"And he made a molten sea, ten cubits from the one brim to the other: it was round all about, and his height was five cubits; and a line of thirty cubits did compass it round about."

For many purposes an approximate number is all that is wanted, and the accurate number, even if we had it, would be of no more use than the approximate one.

Further, we can often find out *how near* the approximate figure is to the accurate one, and can, if we wish, make the approximation progressively nearer and nearer to the accurate number—but this makes the calculation longer. The nature of the problem usually indicates how close the approximation should be. We will examine in some detail how the limits of accuracy may be set.

Units—Large and Small Quantities.

In all applications of mathematics, it is of the utmost importance to select with care the *unit* we use, and to be perfectly clear what it is. A mere number is of little interest, as it only stands for an abstract idea until a unit is attached to it. When that has been done, it stands for something specific. We choose the unit from the following considerations.

1. In measuring the different quantities, we try to use the same range of numbers as we use in the affairs of everyday life. By doing this we can form a better mental picture of the quantities. The unit settles the scale of the numerical picture. For instance, it is difficult to form a clear idea of the size of a field said to be 16,187,425,688 sq. mm. It is much better to say it is 4 ac. We know what a piece of ground the size of an acre is like, and we can picture what 4 ac. would be like.

2. It saves much trouble in long calculations to write numbers consisting of few figures.

3. We try to keep the arithmetic as simple as possible (see p. 20).

The thing to which our numbers refer being given, and the unit chosen, our numbers have become definite, and our sense of proportion tells us which numbers are large and which small. We shall work out an example suggested by Portia's speech in the Trial Scene, where she checkmates Shylock. Portia lays down with full mathematical rigour the conditions under which Shylock can have his bond. It will be instructive to consider the passage almost word for word, as it will bring out some of the most important ideas of higher mathematics.

> *Portia.* Therefore prepare thee to cut off the flesh,
> Shed thou no blood, nor cut thou less nor more
> But just a pound of flesh: if thou tak'st more
> Or less than a just pound, be it but so much
> As makes it light or heavy in the substance,
> Or the division of the twentieth part
> Of one poor scruple, nay if the scale do turn
> But in the estimation of a hair,
> Thou diest, and all thy goods are confiscate.

Shylock is to have a quantity of flesh, so the thing we are talking about is "flesh".

In the bond, a mathematically exact amount of flesh is specified. "The words expressly are 'a pound of flesh'," Portia has just told the Court. She repeats it here—"*just* a pound of flesh".

We will take the pound as our "unit". The number which measures the amount of flesh Shylock is entitled to, in terms of this unit, is 1.

Now Portia sets the Jew the pretty problem of cutting off one pound of flesh, and incidentally of doing it without shedding blood. Apart from the difficulties of bloodless surgery, the first condition is hard enough to fulfil. She sees that Shylock probably cannot cut off "just a pound", and that what he actually cuts off may be more or less than one pound, and she gives him a margin to work on.

Bearing in mind the claims the Jew has made, her sense of the fitness of things leads her to set up two arbitrary standards of approximation or tolerance. If the *difference* in the weight of the flesh actually cut by the Jew and the amount he is entitled to cut does not exceed these standards, she will overlook the error in cutting and consider the Jew to have fulfilled this part of the conditions; in other words, she will regard the difference as of no importance and one which can be ignored.

The amount actually cut may be "less or more", so that if x stands for the actual amount cut in pounds, $(x - 1)$ will be *positive* if the Jew cuts more and *negative* if less. We are only concerned with the difference—we do not mind whether it is one way or the other, i.e. we take

$$(x - 1) \text{ when } x > 1,$$
$$\text{or } (1 - x) \text{ when } x < 1,$$

and write $(x \sim 1)$ for the difference taken positively. The same thing is also often written

$$|\, x - 1 \,| .$$

Take the first standard set—an amount of flesh weighing "the twentieth part of one poor scruple". This amount is $\frac{1}{7000}$ lb. If then x stands for the actual weight of flesh cut, in pounds, the difference between x and 1, taken positively, must be less than $\frac{1}{7000}$.

The conditions can be written

$$(x \sim 1) < \tfrac{1}{7000}. \qquad \text{.....................(3)}$$

If x satisfies (3), Shylock wins; if not, Portia wins.

But the second test is more severe and less specific —"the estimation of a hair". Hairs vary in weight, but, to give the Jew the benefit, we will take a good long fat hair weighing say $\frac{1}{10}$th part of a grain. This is $\frac{1}{10}$th part of the first standard, so our condition runs

$$(x \sim 1) < \frac{1}{70000}. \quad \dots\dots\dots\dots\dots\dots(4)$$

The Jew cannot hope to cut so exactly as this standard requires. He might be sure of his skill to 1 oz., but to "the estimation of a hair" is out of the question. In other words, the degree of accuracy asked for is higher than the degree practically attainable.

The allowable errors assigned by Portia were both "small", the second, of course, being smaller than the first. We mean by this that a piece of flesh weighing only $\frac{1}{70000}$ lb. is small enough, in the circumstances, to be ignored, and we regard $\frac{1}{70000}$ lb. as a small quantity of flesh.

When we describe a quantity as "small" we mean that it is small *enough* for some purpose we are thinking of; not that it is necessarily small in itself.

Portia's purpose was to set limits to the margin allowed to the Jew, and she decided that $\frac{1}{70000}$ lb. was, under the circumstances, small enough to be ignored.

Similarly, if we were measuring lines with a microscope we might say that a length $\frac{1}{10}$ in. was *large*—meaning that it is large enough to be measured without a microscope.

Again, consider a length $\frac{1}{100}$ in. The area represented by its square is $\frac{1}{10000}$ sq. in., the volume represented by its cube, $\frac{1}{1000000}$ c. in., and so on, and if we say that $\frac{1}{100}$ in. is "small" we may mean that it is small enough to permit of the quantities represented by its powers, $(\frac{1}{100})^2$, $(\frac{1}{100})^3$, &c., being neglected in our calculations. We shall see an example of this use of "small" later—it is the most common use of the word in this subject.

We can regard the quantity x as small when

$$|x| < \kappa,$$

where κ is some specially chosen positive quantity which we decide to regard as our standard of smallness, or as our *standard of approximation*.

Small Quantities.

Consider the finite geometrical series

$$1 + r + r^2 + \ldots + r^{n-1}, \qquad \text{.........(5)}$$

where r is a positive fraction and n a positive integer.

We will suppose r is $\frac{1}{10}$ and n is 10, so that the series is

$$1 + \frac{1}{10} + \frac{1}{100} + \ldots + \frac{1}{10^9}.$$

The series has 10 terms, the last of which is $\frac{1}{10^9}$.

The second term is $\frac{1}{10}$ of the first, and, for some purposes, the second term may measure a quantity, *small* relative to that measured by the first term, unity.

The third term is then small relative to the second, since it is $\frac{1}{10}$ of the second, and is as small relatively to the second as the second is relatively to the first. It is, however, far smaller *relatively to the first* than to the second, because it involves the *square* of the "small quantity", $\frac{1}{10}$.

We may call it a small quantity of the *second* order. Similarly, the fourth term, $\frac{1}{1000}$, involves the cube of the small quantity, $\frac{1}{10}$, and is called a small quantity of the third order, and so on for the remaining terms. Our series, then, goes up to a small quantity of the ninth order. In the series (5) if r is "small" relative to unity,

r is called a small quantity of the 1st order,
r^2 " " " 2nd "
r^3 " " " 3rd "

and so on. Whether r *is* "small" compared to unity

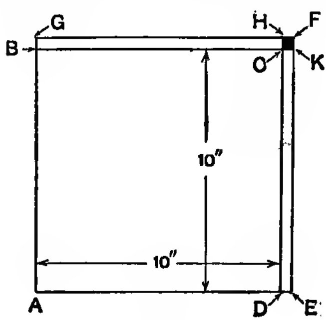


Fig. 3

or not is a question which depends on the particular application we are considering.

The importance of this idea of "orders of smallness" will be seen from an example. Consider the expansion of a square plate of iron by heat.

Let the side of the square plate measure 10 in. at 0° C.

At 10° C. its sides measure 10·00123 in. (fig. 3).

The area of the plate at 10° C. can be very simply calculated with accuracy. It is

$$(10{\cdot}00123)^2 = 100{\cdot}0246015129 \text{ sq. in.}$$

For most purposes, we do not need as many as 13 significant figures in the calculation. It may be sufficient to be correct to say 7 significant figures.

The area of the expanded plate, correct to 7 significant figures, can be calculated by the following method, which depends on the use of "small quantities", thus:

In fig. 3, the area of the expanded plate is AGFE.

The original area is ABCD, hence

area AGFE

$$= \text{area ABCD} + 2\,(\text{area CKED}) + \text{area HFKC},$$

i.e. area AGFE

$$= [10^2 + 2(10 \times 0{\cdot}00123) + (0{\cdot}00123)^2] \text{ sq. in.}$$
$$= [100 + 20 \times 0{\cdot}00123 + (0{\cdot}00123)^2] \text{ sq. in.}$$
$$= [100 + 0{\cdot}0246 + 0{\cdot}000001513] \text{ sq. in.}$$

We have thus expressed the area of the expanded plate as the sum of a series of three terms.

The second term depends on 0·00123, and the third term on $(0{\cdot}00123)^2$. 0.00123 is about $\frac{1}{80000}$th part of the first term, 100, and may legitimately be regarded as "small", relatively.

The second term is therefore a small quantity of the first order, as it depends on the first power of the small quantity (0·00123); while the third term is a small quantity of the second order. It is evident that the third term can be neglected in calculating the area of the expanded plate to 7 significant figures, for 7 significant figures take us only to the fourth decimal place, and the third term contributes nothing before the sixth place of decimals. It arises from the little square HFKC, and is $\frac{1}{66000000}$th part of the first term.

The area of the expanded plate can therefore be obtained as accurately as need be by rejecting the term of the second order of smallness, and retaining only the term of the first order.

The result is 100·0246 sq. in.

Provided the small quantity of the lowest order which we retain is small *enough*, we can neglect all the small quantities of higher order which occur.

To put this calculation into the language of inequalities:

We wished to calculate the area correct to 7 significant figures. The area must be greater than 100 sq. in. and less than 121 sq. in., hence 7 significant figures take us to four places of decimals.

We can therefore neglect areas less than $\frac{5}{100000}$ sq. in.

The true area is

$$[100 + 0{\cdot}0246 + (0{\cdot}00123)^2] \text{ sq. in.}$$

The approximate area, obtained by rejecting the last term, is 100·0246 sq. in.

The numerical difference in these areas is

$$(0{\cdot}00123)^2 \text{ sq. in.}$$

$$\text{Now } (0{\cdot}00123)^2 < \frac{5}{100000}.$$

$\therefore$ $(0{\cdot}00123)^2$ can be neglected in calculating the area.

We cannot neglect the area corresponding to the small quantity of the first order, as this area is

$$20 \times 0{\cdot}00123 = 0{\cdot}0246 \text{ sq. in.},$$

and

$$0{\cdot}0246 \text{ is not less than } \frac{5}{100000}.$$

If we were satisfied with 4 significant figures, we could neglect this term as well as the second, as 4 significant figures require only one decimal place, and hence we could neglect areas less than $\frac{5}{100}$ sq. in., and

$$\{0{\cdot}0246 + (0{\cdot}00123)^2\} < \tfrac{5}{100}.$$

In this case the expansion would not affect the area of the square plate by amounts within the degree of accuracy required.

Expansion of a Cube.

Small quantities of the third order arise in the problem of the expansion of a cube. Suppose the

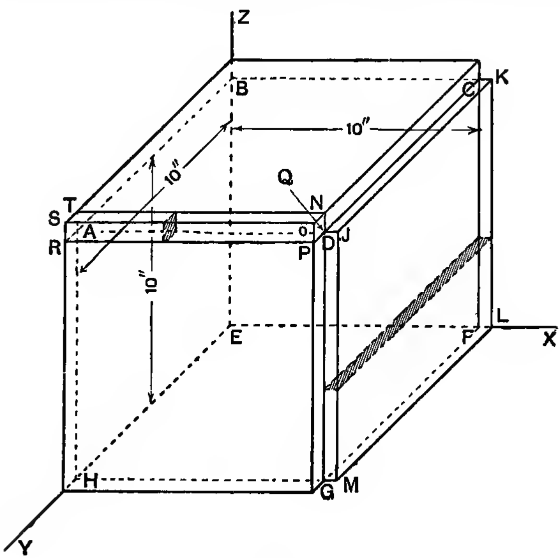


Fig. 4

side of the cube measures 10 in. at 0° C. At 10° C. its side measures 10·00123 in. Fig. 4 illustrates this case. The volume of the expanded cube is

$$(10{\cdot}00123)^3 = 1000{\cdot}369045388860867 \text{ c. in.};$$

but we do not need 19 significant figures for any practical purpose: 7 figures will be quite enough, as in the case of the square.

The expanded cube is got from the original one by making certain additions:

1. A slab like CDGFKJML (fig. 4) is placed on each of the three faces of the cube ABCD, CDGF, and ADGH.

The size of each of these slabs in cubic inches is

$$10^2 \times 0{\cdot}00123 = 0{\cdot}123.$$

The three slabs therefore contribute

$$3 \times 0{\cdot}123 = 0{\cdot}369 \text{ c. in.}$$

2. Three small prisms such as DPONARST will be required to fill up the corners left at the ends of the slabs.

The size of each of these in cubic inches is

$$0{\cdot}00123^2 \times 10 = 0{\cdot}000015129.$$

The three contribute

$$3 \times 0{\cdot}000015129 = 0{\cdot}000045387 \text{ c. in.}$$

3. A tiny cube will be required to fill up the corner at Q. The size of this cube is

$$(0{\cdot}00123)^3 = 0{\cdot}000000001860867 \text{ c. in.}$$

The whole growth is the sum of these items, and the volume (V′) of the expanded cube is the original volume plus these growths, i.e., in cubic inches,

$$\begin{aligned} V' &= [1000 + (3 \times 10^2 \times 0{\cdot}00123) \\ &\qquad + \{3 \times 10 \times (0{\cdot}00123)^2\} + (0{\cdot}00123)^3] \\ &= [1000 + 0{\cdot}369 + 0{\cdot}000045387 \\ &\qquad + 0{\cdot}000000001860867]. \end{aligned}$$

The second term depends on 0·00123, the third on $(0{\cdot}00123)^2$, and the fourth on $(0{\cdot}00123)^3$.

We can consider 0·00123 a small quantity just as in the case of the square. The second term is therefore a small quantity of the first order, the third of the second order, and the last of the third order. It is evident that this is the order of importance of the quantities and the order in which they should be taken to get increasing accuracy. The three slabs (first order terms) contribute most to the growth, the three prisms the next amount, while the little cube contributes the least.

We need only retain the first order term to get the 7 significant figures we need, i.e.

1000·369 c. in. is the desired result.

Note how rapidly the terms of higher order of smallness decrease in importance.

The Changes in the Values of Functions.

Suppose $y = f(x)$.

It is important to find how changes in the value of x affect the value of y.

We may illustrate the general method of dealing with this question by applying a more symbolic treatment to the two practical problems just considered.

(1) Find the increment of area when the sides of a square metal plate expand by a given amount. Suppose x stands for the length of the edge of the plate, and h for the linear expansion (DE or BG, see fig. 3), and y for the area of the plate before expansion, then, if y' is the area of the expanded plate,

$$y' = (x + h)^2$$
$$\text{and } (y' - y) = (x + h)^2 - x^2$$
$$= 2xh + h^2. \qquad \ldots\ldots\ldots\ldots\ldots (6)$$

$\therefore$ the *increase* in area is $2xh + h^2$.

A *finite* increase in x is often written Δx. This does not mean $\Delta \times x$, but is a *single symbol* standing for "a finite increase in" x. Similarly, a finite increase in y is written Δy.

In this notation we get

$$\Delta y = 2x\Delta x + (\Delta x)^2. \qquad \text{.................}(7)$$

This equation gives the information we require, viz. the increment in the area (y) arising from a given linear expansion (Δx) of the side (x).

The formula has been calculated for an expanding plate, but it is clear that it is true whatever x and y may stand for, provided $y = x^2$.

The formula gives the increase in x^2 when x increases by Δx, whatever values x and Δx may have.

WORKED EXAMPLE

Suppose $x = 9$ and $\Delta x = 2$, then

$$\begin{aligned} \Delta y &= 2 \times 9 \times 2 + 2^2 \text{ on substituting in (7)} \\ &= 36 + 4 \\ &= 40. \end{aligned}$$

Check. When $x = 9$ $\quad x^2 = 81$

$x = 11$ $\quad x^2 = 121$

Diff. $= 40.$

$\therefore$ growth in $x^2 = 40$ when x increases from 9 to 11.

The Ratio $\Delta y/\Delta x$, when $y = x^2$.

If y stands for the area of the plate in square inches, and x for the length of the side in inches,

$$\Delta y = 2x\Delta x + (\Delta x)^2 \text{ by (7)}. \qquad \text{..................}(8)$$

Consider what happens when we cool the plate from say 10° C., towards the original temperature, 0° C. The linear expansion becomes smaller and smaller, and hence Δx becomes smaller and smaller. When the rise of temperature is 10° C., Δx is 0·00123; when the rise is 1° C., Δx is 0·000123; when the rise is $\frac{1}{1000}$° C., Δx is 0·000000123; and so on. But there must be *some* rise of temperature, otherwise the problem is not one of the expansion of a warmed plate at all. Hence the value, zero, of Δx does not interest us.

We will suppose that Δx can be as small as we like, but not zero. When Δx is not zero, we can divide equation (8) by Δx thus:

$$\frac{\Delta y}{\Delta x} = 2x + \Delta x. \quad \ldots\ldots\ldots\ldots\ldots(9)$$

Now, by taking Δx small enough, we can bring $2x + \Delta x$ as near to $2x$ as we please.

For instance, $(2x + \Delta x)$ differs from $2x$ by less than 1 billionth, when $\Delta x <$ 1 billionth, and so on.

We express these ideas by saying that the *limit* of $(2x + \Delta x)$, as Δx *approaches* zero, is $2x$, and we write

$$\underset{\Delta x \to 0}{\mathrm{Lt}} (2x + \Delta x) = 2x.$$

The symbol $\to$ reads "*approaches*".

Reverting now to (9), we get

$$\underset{\Delta x \to 0}{\mathrm{Lt}} \left(\frac{\Delta y}{\Delta x}\right) = \underset{\Delta x \to 0}{\mathrm{Lt}} (2x + \Delta x) = 2x. \ldots(10)$$

The fraction $\Delta y/\Delta x$ therefore tends to the limit $2x$, as Δx approaches zero.

The limiting value of $\Delta y/\Delta x$ as $\Delta x \to 0$ is denoted by $\left(\frac{d}{dx}\right)y$.

The symbol $\frac{d}{dx}$ does not mean $d \div dx$, and is not a symbol for *a number* at all. It is *a symbol standing for an operation*, which, when applied to y, is the operation of taking the limit of the ratio of Δy to Δx, when $\Delta x \rightarrow 0$, so that

$$\left(\frac{d}{dx}\right)y = \underset{\Delta x \rightarrow 0}{\text{Lt}}\left(\frac{\Delta y}{\Delta x}\right). \quad \ldots\ldots\ldots\ldots(11)$$

And if $y = x^2$

$$\left(\frac{d}{dx}\right)y = 2x. \quad \ldots\ldots\ldots\ldots\ldots\ldots(12)$$

The symbol $\frac{d}{dx}$ may be compared with $\surd$. This sign is a symbol of operation. $\sqrt{x}$ does not mean $\surd \times x$, but it means the result of the operation of extracting the square root of x.

$\left(\frac{d}{dx}\right)y$ is usually printed $\frac{dy}{dx}$ for convenience, but when it occurs in this form, it *must not be mistaken for a fraction* $(dy \div dx)$.

Summarizing the notation, we have

$$\left(\frac{d}{dx}\right)y = \frac{dy}{dx} = \underset{\Delta x \rightarrow 0}{\text{Lt}}\left(\frac{\Delta y}{\Delta x}\right); \quad \ldots\ldots\ldots(13)$$

or, putting $f(x)$ for y if $y = f(x)$,

$$\left(\frac{d}{dx}\right)f(x) = \frac{df(x)}{dx} = \underset{\Delta x \rightarrow 0}{\text{Lt}}\left(\frac{\Delta f(x)}{\Delta x}\right). \quad \ldots(14)$$

It is important that the reader should see clearly the necessity for this treatment by limiting values.

For instance, it might be asked, why not put $\Delta x = 0$ in the equation

$$\frac{\Delta y}{\Delta x} = 2x + \Delta x,$$

and get $\frac{\Delta y}{\Delta x} = 2x$ and avoid using the idea of the limit?

The answer is that Δx cannot be both something and nothing; it must be something *or* nothing.

If it is nothing, there is no change in x, so $\Delta y/\Delta x$ is meaningless; in fact, the fraction cannot arise.

It therefore must be something, but it can *approach* zero as nearly as we like. This possibility suggests our trying to find a number l such that

$$\frac{\Delta y}{\Delta x} \rightarrow l$$

$$\text{as } \Delta x \rightarrow 0.$$

This number l, if we can find it, is called *a limit*, and we express the fact that the limit l exists (if it does) by writing

$$\operatorname*{Lt}_{\Delta x \rightarrow 0}\left(\frac{\Delta y}{\Delta x}\right) = l = \frac{dy}{dx}. \quad \ldots\ldots\ldots\ldots\ldots\ldots(15)$$

as a matter of notation.

(2) The symbolic treatment of the second of the two practical problems, viz. to find the volumetric expansion of a metal cube due to a given expansion of the side, will serve to emphasize the foregoing fundamental principles.

Let x stand for the length of the edge of the cube in inches, and y for the volume in cubic inches.

Let Δx stand for the expansion of the side in inches.

Then, if y is the volume of the expanded cube,

$$\begin{aligned} y' &= (x + \Delta x)^3. \\ \therefore y' - y &= (x + \Delta x)^3 - x^3. \\ \therefore \Delta y &= 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3. \quad \ldots.(16) \end{aligned}$$

This equation gives us the result required, viz. the volumetric expansion in terms of the linear expansion.

EXAMPLES

1. *The side of a cube of brass expands from 12 in. at 0° C. to 12(1 + 0·000187) in. at 10° C. Calculate the increase in its volume, and the ratio of increase in volume to increase in length of side. Work to 4 significant figures.*

2. *Find the change in the value of y ($= x^3$) when x changes from 9 to 11, and check the result given by the formula* (16) *by a direct calculation.*

To find dy/dx when $y = x^3$.

We have by (16)

$$\Delta y = 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 \ldots\ldots (17)$$

$$\therefore \frac{\Delta y}{\Delta x} = 3x^2 + 3x(\Delta x) + (\Delta x)^2.$$

$$\frac{dy}{dx} = \underset{\Delta x \to 0}{\mathrm{Lt}} \left(\frac{\Delta y}{\Delta x}\right),$$

hence we require the limit of $\frac{\Delta y}{\Delta x}$ as $\Delta x \to 0$.

$3x(\Delta x)$ may be made as small as we please by taking Δx small enough.

$$\therefore 3x(\Delta x) \to 0 \text{ as } (\Delta x) \to 0.$$

Similarly, $(\Delta x)^2 \to 0$ as $(\Delta x) \to 0$.

$$\therefore (3x^2 + 3x\Delta x + (\Delta x)^2) \to 3x^2 \text{ as } \Delta x \to 0;$$

$$\text{i.e. } \frac{dy}{dx} = \underset{\Delta x \to 0}{\mathrm{Lt}} \left(\frac{\Delta y}{\Delta x}\right) = 3x^2, \ldots\ldots (18)$$

when $y = x^3$.

To find $\frac{d}{dx}(x^n)$; n a positive integer.

We have seen, on p. 40, that, if $y = f(x)$,

$$\frac{dy}{dx} = \frac{df(x)}{dx} = \underset{\Delta x \to 0}{\mathrm{Lt}} \left(\frac{\Delta f(x)}{\Delta x}\right),$$

and for two particular functions of x, viz. x^2 and x^3, we have found the respective limiting values to be $2x$ and $3x^2$. That is

$$\frac{dy}{dx} = 2x \text{ when } y = x^2,$$

$$\text{and } \frac{dy}{dx} = 3x^2 \text{ when } y = x^3.$$

We will now proceed to find the value of

$$\frac{dy}{dx} \text{ when } y = x^n,$$

n being a positive integer. The algebra will be a little long, and it will shorten the expressions to write h, instead of Δx, for the increase in x.

Multiply

$a^2 + ab + b^2$ by $a - b$. Result $a^3 - b^3$;
$a^3 + a^2b + ab^2 + b^3$ by $a - b$. Result $a^4 - b^4$.

In general,

$$(a - b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \dots + b^{n-1}) = a^n - b^n. \quad \text{.................}(19)$$

This identity can be easily proved by direct multiplication.

It is true for all values of a and b.

Rewrite it

$$\frac{a^n - b^n}{a - b} = a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \dots + b^{n-1}. \quad \text{.....}(20)$$

There are n terms on the right-hand side.

$$\text{Put } a = x + h \text{ and } b = x.$$
$$\text{Then } a - b = h,$$
$$\text{and } a^n - b^n = (x + h)^n - x^n.$$
$$\therefore \frac{a^n - b^n}{a - b} = \frac{(x + h)^n - x^n}{h}. \quad \text{.......}(21)$$

Substituting $a = x + h$ and $b = x$ in the right-hand side of (20) we get

$$(x + h)^{n-1} + (x + h)^{n-2}x + (x + h)^{n-3}x^2 + \dots + x^{n-1}.$$

Now, $(x + h)^n - x^n$ is the increase which takes place in y when the independent variable x changes by the amount h, i.e. by Δx. Therefore

$$\frac{\Delta y}{\Delta x} = \left[\frac{(x+h)^n - x^n}{h}\right].$$

The limiting value of this expression, as $h \rightarrow 0$, is by definition, the value of $\frac{d}{dx}(x^n)$. The limiting value we require can be easily found.

$$\begin{aligned} \text{When } h \rightarrow 0,\ (x+h)^{n-1} &\rightarrow x^{n-1} \\ (x+h)^{n-2}x &\rightarrow x^{n-1} \\ (x+h)^{n-3}x^2 &\rightarrow x^{n-1} \\ \cdot\ \cdot\ \cdot\ \cdot &\rightarrow \cdot\ \cdot\ \cdot \\ x^{n-1} &\rightarrow x^{n-1} \end{aligned}$$

Adding,

$$(x+h)^{n-1} + (x+h)^{n-2}x + \ldots + x^{n-1} \rightarrow n \times x^{n-1},$$

as there are n terms.

$$\therefore \operatorname*{Lt}_{h \rightarrow 0}\left[\frac{(x+h)^n - x^n}{h}\right] = nx^{n-1},$$

$$\text{i.e. } \frac{d}{dx}(x^n) = nx^{n-1} \text{......(22)}$$

Derivative of a Function.

If $y = f(x)$, and $\frac{dy}{dx} = \psi(x)$, $\psi(x)$ is called the *derivative* of $f(x)$.

Equation (22) expresses an important rule which can now be put in words.

Rule I.—The derivative of x^n is n times x^{n-1} when n is a positive integer. The rule is true for all values of x.

Thus, the derivative of x^2 is $2x$, that of x^3 is $3x^2$, and so on.

Exercise 1

1. Draw a circle of radius $2\frac{1}{2}$ in. Divide a diameter AB into 10 equal parts, and draw lines through the points of division, cutting the circumference of the circle and perpendicular to the diameter chosen. By the method shown on pages 13, 14, find upper and lower limits for the area of the circle, and express the difference in these limits as a percentage of the lower limit.

2. Repeat (1), dividing AB into 20 equal parts. Notice that by making the strips half as wide as in (1), the area is determined within much closer limits. Find these limits, and compare them with the true area, calculated from πr^2.

3. If $y = x^2$, calculate the increase in y corresponding to an increase Δx in x, when $x = 6$, and also the value of $\frac{\Delta y}{\Delta x}$. Calculate the values of Δy and $\frac{\Delta y}{\Delta x}$ when $\Delta x = 0{\cdot}1$, $0{\cdot}01$, $0{\cdot}001$, &c., and fill in the blank spaces in the table below, to show the way in which $\frac{\Delta y}{\Delta x}$ approaches its limiting value 12, as Δx approaches zero.

Δx	0·1	0·01	0·001	0·0001
Δy				
$\frac{\Delta y}{\Delta x}$				

4. If $y = x^3$ find Δy and $\frac{\Delta y}{\Delta x}$ corresponding to an increase Δx in x when $x = 8$, and calculate their values when $\Delta x = 0{\cdot}1$, $0{\cdot}01$, $0{\cdot}001$. Fill in the blank spaces in a table similar to that above, to show the way in which $\frac{\Delta y}{\Delta x}$ approaches its limiting value, 192, as $\Delta x \rightarrow 0$.

5. Find from first principles the derivative of x^4.

6. Differentiate:

$$\text{(i) } y = x^5.$$
$$\text{(ii) } y = x^8.$$
$$\text{(iii) } y = x^{11}.$$

7. A square plate of brass is expanding by heat. When the length of its side is 7 in. it increases by ·002 in. in a second. Give an approximate value for the corresponding rate of increase in the area of the plate.

CHAPTER II

Geometrical and Mechanical Meaning of a Derivative

"Nothing can be more fatal to progress than a too confident reliance on mathematical symbols; for the student is only too apt to take the easier course, and consider the *formula* and not the *fact* as the physical reality."—KELVIN AND TAIT (*Natural Philosophy*).

In Chapter I, we found that if $y = f(x)$, we are sometimes able to find a limit l to which the fraction $\frac{\Delta y}{\Delta x}$ tends in value as Δx approaches zero. We express this by writing

$$\underset{\Delta x \longrightarrow 0}{\text{Lt}} \left(\frac{\Delta y}{\Delta x}\right) = l = \frac{dy}{dx}.$$

This limit we called the derivative of $f(x)$.

In this chapter we shall try to find out the physical significance of this *limit*.

Gradient or Slope of a Straight Line.

Let O′P, fig. 1, be any straight line, and let Ox, Oy be any pair of rectangular axes. Choose any point P in this line, and let (x, y) be the co-ordinates of P.

Then OM $= x$ and MP $= y$.

Let the straight line intersect the Oy axis in the point O′ whose co-ordinates are (0, c).

Through P draw MP parallel to Oy, and through

O′ draw O′N parallel to Ox, N being the point of intersection of this line with MP.

Then the ratio of NP : O′N is called the *gradient* of the line O′P. If we suppose Ox to be a horizontal

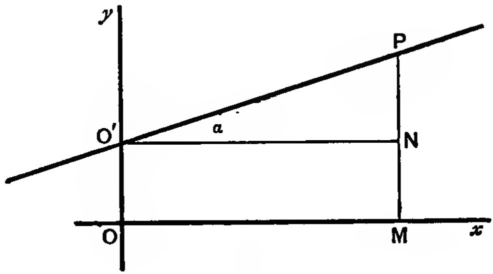


Fig. 1

line, this definition of "gradient" agrees with the ordinary meaning of the word. Let α be the angle of inclination of O′P to O′N, then

$$\frac{\text{NP}}{\text{ON}} = \tan\alpha = \text{gradient of the given straight line.}$$

Gradient or Slope of a Curve—Definition.

The gradient of a curve, at a given point on it, is defined to be the gradient of the tangent to the curve at the point in question.

Equation of a Straight Line.

Referring to fig. 1, we see that, if P is any point on the straight line,

$$\begin{aligned} \text{MP} &= \text{MN} + \text{NP} \\ &= \text{OO}' + \text{NP} \\ &= \text{OO}' + \text{OM} \cdot \tan\alpha. \\ \therefore y &= x \tan\alpha + c. \end{aligned}$$

Putting m for $\tan\alpha$,

$y = mx + c$, where m and c are constants. (1)

This is the equation of a straight line, and it is an equation of the first degree in x and y.

This equation can be easily transferred into any other form in which a linear equation may be written, thus:

$$y = mx + c.$$

Multiply by B throughout, then

$$By = Bmx + Bc.$$

Put $(-A)$ for the constant Bm, and $-C$ for Bc, and we get

$$By = -Ax - C.$$

$$\text{i.e. } Ax + By + C = 0.$$

Although $Ax + By + C = 0$ appears to have three independent constants, it really only has two, as we can reduce the coefficient of y to unity by dividing the equation by B.

Parallel Straight Lines.

Parallel straight lines are those having the same gradient, i.e. the same m, hence if

$y = mx + c$ represents a given straight line,
$y = mx + c'$,, a parallel straight line,

where c and c' are different numbers.

When the straight line is parallel to Ox, its equation becomes

$$y = c,$$

$$\text{for } m = \tan\alpha = \tan 0° = 0,$$

and when parallel to Oy, the equation is

$$x = d.$$

In these equations c and d are constants.

WORKED EXAMPLE

Find the equation of a straight line passing through the points $(-3, 4)$ *and* $(2, -1)$.

Let the equation be

$$y = mx + c.$$

Then, substituting the given values of the co-ordinates of the two given points for x and y, we get

$$4 = -3m + c$$

$$\text{and } -1 = 2m + c.$$

$$\text{Subtracting, } 5 = -5m.$$

$$\therefore m = -1,$$

$$\text{and } c = 1;$$

therefore the equation of the straight line is

$$y = -x + 1,$$

$$\text{or } x + y = 1.$$

Geometrical Meaning of $\frac{dy}{dx}$.

Let AB be a portion of a continuous graph (fig. 2).

Let $y = \phi(x)$ be the equation to this graph.

Let P be any point (x, y) on the curve, which we choose at pleasure, and PQ a straight line through P, cutting the curve at Q.

We will suppose a tangent to be drawn to the curve AB at any point, P, on AB. Let PT be this tangent line at P, and let $\angle TPR = \psi$, where PR is parallel to Ox.

Further, suppose PT is not parallel to Oy.

So long as P and Q are distinct points, the line PQ cuts the curve at P and Q and PQ is not a tangent; further the angle QPR $> \psi$.

Let the line QPM rotate about P in the clockwise direction, and the line M′PQ′ about P in the counter-clockwise direction, subject to the restriction that

Q and Q′ can approach P as near as desired, but must not coincide with P.

It is evident that, as Q and Q′ approach P, the lines QPM and M′PQ′ will swing round towards a

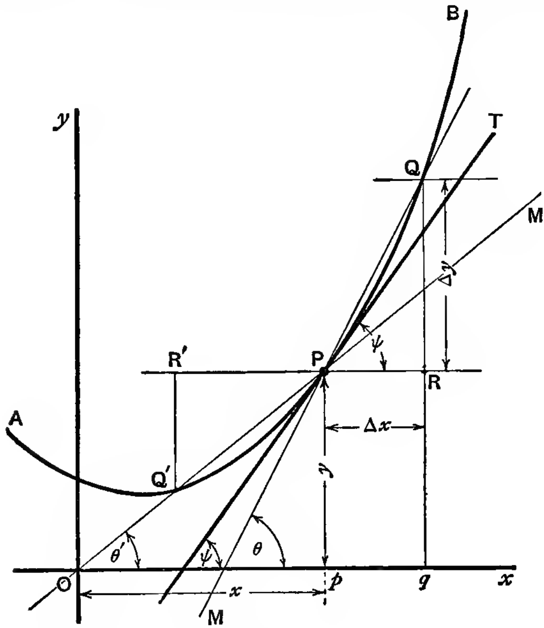


Fig. 2

limiting position, and that this limiting position will be the tangent line at P, for QPM cannot *pass* the tangent position, otherwise Q must become coincident with P in the process; and, similarly, M′PQ′ cannot *pass* the tangent position, but the lines QPM and M′PQ′ can lie as near together as we like by taking Q and Q′ sufficiently near to P.

It is evident therefore that, as Q (or Q′) → P,

$$\theta \to \psi$$
$$\text{and } \theta' \to \psi$$

where ψ is the inclination of the tangent to the Ox direction; i.e.

$$\tan\theta \to \tan\psi \text{ as Q} \to \text{P.}$$

But

$$\tan\theta = \frac{\Delta y}{\Delta x} \text{ for } \tan\theta = \frac{\text{RQ}}{\text{PR}},$$

and RQ is the change in the value of y brought about by a change PR in the value of x.

$$\text{As Q} \to \text{P}, \ \Delta x \to 0,$$
$$\therefore \frac{\Delta y}{\Delta x} \to \tan\psi \text{ as } \Delta x \to 0,$$

i.e. $\tan\psi$ is the limiting value of

$$\frac{\Delta y}{\Delta x} \text{ as } \Delta x \to 0, \text{ at P.}$$

$$\text{i.e. } \left[\frac{dy}{dx}\right]_{x = \text{O}p} = \tan\psi$$

where $\left[\frac{dy}{dx}\right]_{x = \text{O}p}$ is the value of $\frac{dy}{dx}$ at P and ψ is the angle of inclination of the tangent at P, to the Ox axis.

On the understanding that both $\frac{dy}{dx}$ and ψ are to be measured at P,

$$\frac{dy}{dx} = \tan\psi. \quad \text{..................(2)}$$

The reader is advised to draw out fig. 2 on a large scale and test, by drawing, the fact that θ and $\theta' \to \psi$ as Q and Q′ swing towards P.

Equation (2) gives us a clear idea of the geometrical meaning of dy/dx. It is the trigonometrical tangent of the angle of inclination of the geometrical tangent at P, to the Ox axis. But this is the gradient of the curve at P, hence

$$\frac{dy}{dx} \text{ or } \frac{d}{dx}f(x)$$

is the measure of the gradient of the curve which is the graph of $y = f(x)$ at the point at which it is calculated.

For example: Let $y = x^2$.

$$\text{Then } \frac{dy}{dx} = 2x.$$

$$\text{At the point } x = 10,\ 2x = 20.$$

$$\therefore \left(\frac{dy}{dx}\right)_{x = 10} = 20.$$

$\therefore$ the slope of the graph of $y = x^2$ at $x = 10$ is 20.

i.e. $\tan \psi = 20$ at $x = 10$.

i.e. $\psi = \text{arc}\tan 20$ at $x = 10$.*

i.e. $\psi = 87°10'$ about, at $x = 10$.

The tangent to the graph of $y = x^2$ which touches the curve at the point P[$x = 10$, $y = 100$] cuts the Ox axis at an angle of 87° 10′.

EXAMPLE

Draw the graph of $y = x^2$ to the scale 1 *in.* = 1 *horizontally* (x *axis*) *and* 1 *in.* = 10 *vertically* (y *axis*). *Mark the point P for which $x = 10$, and therefore $y = 100$. Draw the tangent at P as accurately as you can, and measure the angle at which it cuts Ox, with a protractor. What is this angle? Does it confirm the preceding calculation?*

* This notation—the Continental one—is now becoming common in Britain instead of the older form $\tan^{-1} 20$.

The result we obtained in formula (2) is true for all functions of x the graphs of which are smooth curves,

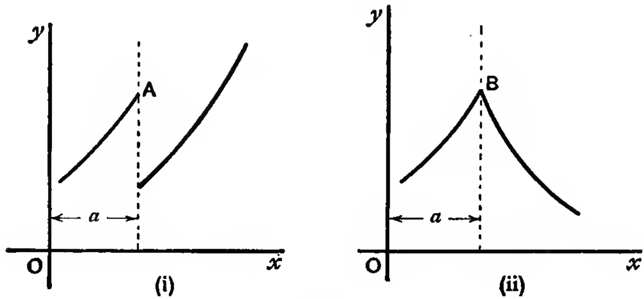


Fig. 3

i.e. are curves free from "breaks" or "kinks". Fig. 3 shows what this statement means.

(i) Shows a "break" at $x = a$.
(ii) Shows a "kink" at $x = a$, but no "break".

Obviously neither of these curves has *one* definite tangent line at the points A and B, and so there is no definite ψ, and consequently equation (2) gives no definite value for $\frac{dy}{dx}$, hence any function having a graph like that shown in fig. 3 (i) or (ii) would not have a single definite *derivative* at $x = a$.

Speed.

A train sometimes runs in such a way that it travels equal distances in equal times. We then describe it as running at a uniform or constant speed.

Let v be the number of miles travelled in any one hour; then, in t hours, the distance travelled is vt

miles, i.e. if s stands for the distance travelled, in miles,

$$s = vt. \qquad\qquad (3)$$

It is evident from (3) that when a body moves with a uniform or constant velocity, the graph of its motion,

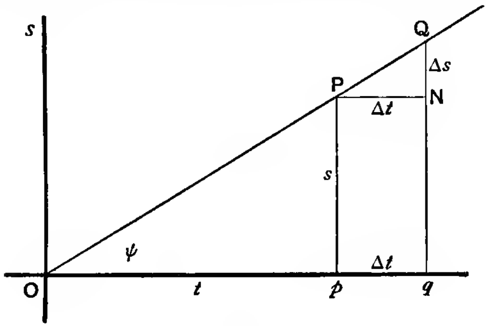


Fig. 4

as a distance-time curve, is a straight line through the origin. The graph is shown in fig. 4.

Consider *any* point P (fig. 4) on the graph.

$$pP = s$$

$$Op = t,$$

$$\text{hence } \frac{pP}{Op} = \frac{s}{t} = v \text{ by equation (3).}$$

$$\text{But } \frac{pP}{Op} = \tan\psi, \text{ if } \psi \text{ is the angle of slope of the line OP,}$$

$$\text{hence } \tan\psi = v.$$

Consider a neighbouring point Q on the graph. Through Q draw the ordinate Qq, and draw PN parallel to Ot, and cutting Qq in N.

$$\text{Then, NQ} = \Delta s$$

$$\text{and PN} = \Delta t,$$

as these lines represent, respectively, the increases of s and t under the condition of constant speed.

$$\text{Now, } \frac{\text{NQ}}{\text{PN}} = \tan \text{Q}\hat{\text{P}}\text{N}$$
$$= \tan\psi$$
$$= v.$$

$$\therefore \frac{\Delta s}{\Delta t} = v = \text{the constant velocity,}$$

no matter over what interval (Δt) the increase (Δs) in s is taken.

Again, $$\tan\psi = \frac{ds}{dt} \text{ by equation (2).}$$

$$\therefore \frac{ds}{dt} = v, \text{ a constant,}$$
$$= \text{the velocity of the train.}$$

In the case of constant speed, we have then

$$v, \text{ the constant speed, } = \frac{\Delta s}{\Delta t} = \frac{ds}{dt} = \frac{s}{t} = \tan\psi. \quad (4)$$

These different ways of expressing the velocity arise from the fact that it is *constant*, and the graph of distance-time, a straight line.

Variable Speed.

It is a fact of everyday life that the speed of a train is not constant, but variable. This is the very kind of obstacle the calculus was invented to overcome. Here, the distances travelled in equal intervals of time are not equal.

Motion in a Straight Line with Variable Velocity.

Suppose a train moves in a straight line from O to P (fig. 5) in t hours where OP corresponds to s miles.

If the point moved with uniform velocity v m.p.h. we should have $s = vt$ or $v = \frac{s}{t}$ mi./hr. This is

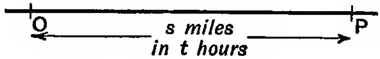


Fig. 5

the average velocity of the point during the t hours of its motion from O to P.

Definition of Average Velocity.

The average velocity of a point during any interval of time is that uniform velocity with which the point would describe the distance travelled in the same time.

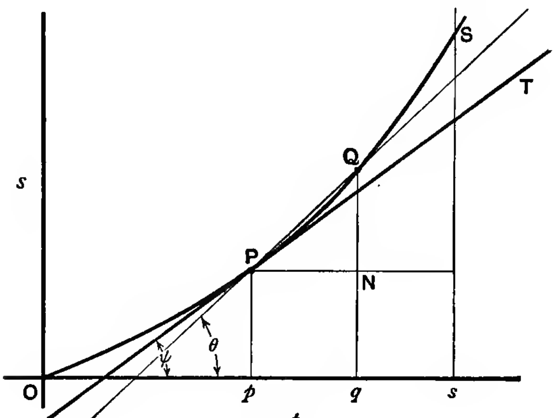


Fig. 6

Instantaneous Velocity.

Suppose the graph of distance-time is not a straight line, but is as shown in fig. 6.

Consider the graph from $t = 0$ to $t = Os$.

At the end of t hours, the train has moved a distance s miles; thus if P corresponds to (s, t),

$$Op = t \text{ and } pP = s.$$

At the end of time $(t + \Delta t)$ hours, the train has moved a distance $(s + \Delta s)$ miles from the start, i.e. if Q represents this state of things,

$$Oq = t + \Delta t \text{ and } qQ = s + \Delta s.$$

Join P, Q by the straight line PQ. If the actual motion between t and $t + \Delta t$ were represented by this straight line PQ, it would be a motion at uniform speed.

Draw PN parallel to Ot to meet the Q ordinate at N. Then the uniform speed corresponding to the straight line PQ is

$$v = \frac{NQ}{PN} = \frac{\Delta s}{\Delta t}.$$

This (uniform) speed, in miles per hour, is the *average* speed between t and $(t + \Delta t)$.

But
$$\frac{\Delta s}{\Delta t} = \tan\theta,$$

where θ is inclination of chord PQ to the Ot direction, hence $\tan\theta$ is the average speed over the interval Δt at P.

We know that, if

$$\frac{\Delta s}{\Delta t} \rightarrow \text{a limit as } \Delta t \rightarrow 0,$$

this limit is $\tan\psi$, where ψ is the angle of slope of the tangent at P, see equation (2).

If we assume that a tangent to the curve can be drawn at any and every point—and it certainly appears as if it could—then $\Delta s/\Delta t$ approaches a limit at each point, and this limit is tan ψ, so that

$$\frac{ds}{dt}, \text{ at P}, = \tan\psi, \text{ at P}.$$

Now, $\frac{\Delta s}{\Delta t}$ is the average speed over the interval of time from t to $(t + \Delta t)$, hence this average speed approaches a limit, denoted by $\frac{ds}{dt}$, as $\Delta t \rightarrow 0$. This limit, $\frac{ds}{dt}$, is called the *instantaneous velocity* at the moment in question, since it is the *velocity* to which the average velocity over an interval of time (Δt) at t tends, as the interval (Δt) approaches zero. Briefly, it is called "*the velocity at t*", and is the velocity the train possesses at that moment. It will be remembered that an average velocity is necessarily a uniform one, by definition, and the limiting velocity is merely the final value, so to speak, of an average velocity, namely, the average velocity over an interval Δt at t. The instantaneous velocity is therefore measured just as a uniform one is, i.e. if the unit of distance is miles and of time, hours, the proper unit for the instantaneous velocity is *miles per hour.* It is numerically the same as the distance the train would travel in the next hour of its journey if its speed were unaltered.

Definition of Instantaneous Velocity.

The instantaneous velocity at a given moment is ds/dt, calculated at that moment, or *the* velocity at time t is ds/dt at that instant.

EXAMPLE

Suppose the distance travelled in a given interval is given by

$$s = t^2$$

where s is in feet, and t in seconds.

Then

$$\frac{ds}{dt} = 2t.$$

Hence the velocity at exactly 10 sec. from the start ($t = 0$) is $2 \times 10 = 20$ ft. per second, i.e. if the moving body moved for the next 1 sec. with its speed unaltered, it would move 20 ft.

The reader is strongly advised to draw, say, a parabola on a large scale. Take Os say 20 cm. and sS 20 cm., the parabola being

$$s = \tfrac{1}{20} t^2.$$

Divide Os into 10 equal parts. Draw in the ordinates and the "PQ" straight lines of fig. 6.

Determine $\frac{\Delta s}{\Delta t}$ for each strip, and plot out a *speed*-time curve on the scale Os = 20 cm., and a convenient scale for "speed". This curve will consist of a series of steps.

Repeat the experiment to find how fine the divisions must be in order that—

(*a*) The straight lines "PQ" may be practically indistinguishable from the curve.

(*b*) The "steps" in the speed-time curve may practically disappear.

In the preceding discussion, we have tried to show that one of the properties of a body in uniform motion, namely, velocity, may be measured, in terms of a suitable unit, by a definite and constant number; also that when the motion is not uniform, we can still regard the body as possessing a definite velocity at each and every instant, though this velocity varies from instant to instant. In this discussion, we started from the only information we had, namely, that the body moved along, and so was at different distances from the starting-point at different times. This information gives a distance-time curve, which, if the motion is continuous, must be a smooth curve.

If we are prepared to take this instantaneous velocity for granted and also to assume that it is a function of time, whose graph is smooth, the problem of connecting this velocity with distance and time becomes very easy. In this case we can assume a *velocity*-time

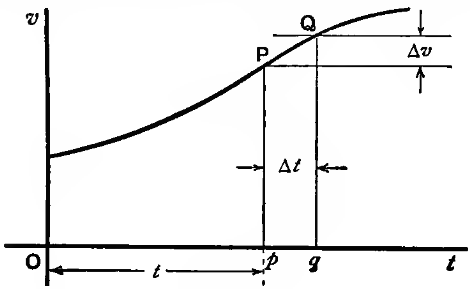


Fig. 7

curve to start with. Before, we started with a distance-time curve.

Let fig. 7 be a graph of such a velocity-time curve where pP, for instance, measures the instantaneous velocity at the instant t, corresponding to Op. Consider a later instant, $t + \Delta t$.

The instantaneous velocity at instant $t + \Delta t$ is qQ.

Let $v = p\text{P} =$ velocity at P,
and $v + \Delta v = q\text{Q} =$,, Q,

and suppose the velocity is increasing with time at P, as in the figure.

Let Δs stand for the increase in s, due to the movement which takes place in the time Δt.

Then $\Delta s > v\Delta t$, for the average velocity during Δt is more than v, and if the velocity during Δt were uniform at v, the distance travelled would be $v\Delta t$.

And $\Delta s < (v + \Delta v)\Delta t$ for similar reasons.

$$\therefore v < \frac{\Delta s}{\Delta t} < v + \Delta v.$$

Now when $\Delta t \rightarrow 0$, $\Delta v \rightarrow 0$, but v does not change since it is the definite velocity at the time t, and though v varies from time to time, it does *not* vary at the *same* instant t, and we are letting Δt change, not t.

$$\therefore \underset{\Delta t \rightarrow 0}{\text{Lt}} (v + \Delta v) = v,$$

and hence

$$\underset{\Delta t \rightarrow 0}{\text{Lt}} \left(\frac{\Delta s}{\Delta t}\right) = v,$$

$$\text{i.e. } \frac{ds}{dt} = v, \quad \dots\dots\dots\dots\dots\dots(5)$$

where both $\frac{ds}{dt}$ and v are measured at the same (and any) point P.

In the figure, v is drawn steadily increasing with t.

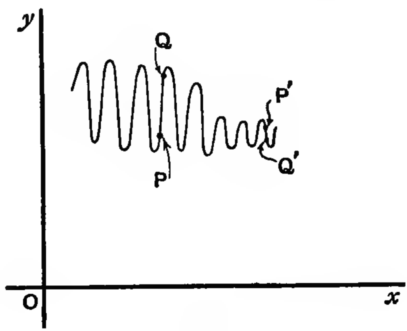


Fig. 8

This is not essential. If the curve is as sketched in fig. 8, we can always take a Q near enough to P to ensure that the ordinate steadily increases (or decreases) from P to Q.

Extension of the Idea of Speed.

We can extend the idea of speed, and its measurement by a *derivative*, by using the word "rate" instead of speed. One thing changes at a *constant rate* with respect to another, when equal changes in the one correspond to equal changes in the other. The amount of the change in the second thing chosen is usually a "unit" amount, and therefore is denoted by unity. The amount of the change of the first thing is then so much of it, *per unit change of the second.*

Let m be the change in the first thing [Y] which corresponds to any unit change in the second thing [X]; then if the change in [X] is X, the corresponding change in [Y] is mX, i.e. if Y stands for the change in [Y]

$$Y = mX,$$

where m is the "rate" in question.

If y and y_0 are the final and initial measures of the first thing [Y], and x, x_0, of the second [X],

$$\begin{aligned} \text{then} \qquad Y &= y - y_0, \\ \text{and} \qquad X &= x - x_0. \\ \therefore y - y_0 &= m(x - x_0). \\ \therefore y &= mx - (mx_0 - y_0) \end{aligned}$$

But $mx_0 - y_0$ is a constant, $-c$, say.

$$\therefore y = mx + c.$$

This is the equation of a straight line, the gradient of which is m, and which cuts the Oy axis at (o, c).

We therefore arrive at a very important result, namely, when one thing changes *at a constant*

***rate* with respect to another, the graph connecting the related quantities of the two things is a straight line.**

This use of the word "rate" has been current since the end of the sixteenth century.

"Six score acres after the rate of 21 foote to every pearche of the sayd acre."—SPENCER, 1596.

It differs only from "speed" in being more general. "Rate" includes "speed", but "speed", since it refers only to distance and time, does not include "rate". The equation of any straight line is

$$y = mx + c, \text{ where } m \text{ and } c \text{ are constants,}$$

$$\therefore \Delta y = m\Delta x.$$

$$\therefore \frac{\Delta y}{\Delta x} = m.$$

$$\therefore \underset{\Delta x \to 0}{\text{Lt}} \left(\frac{\Delta y}{\Delta x}\right) = \underset{\Delta x \to 0}{\text{Lt}} (m)$$

$$\therefore \frac{dy}{dx} = m, \qquad \text{(6)}$$

since $\underset{\Delta x \to 0}{\text{Lt}} (m) = m$, as m is constant.

But m need not be constant. We deal with a rate which varies from point to point on the same lines as a varying speed.

We consider the *average* rate between x and $(x + \Delta x)$, defining the average rate as the *uniform* rate which produces the change in y, corresponding to the chosen change Δx in x. The average rate, *so defined*, is measured by $\Delta y/\Delta x$, just as the average speed is measured by $\Delta s/\Delta t$.

The limit to which $\Delta y/\Delta x$ tends, as $\Delta x \to 0$, is the rate to which the average rate tends, as the interval Δx approaches zero. Hence, if m is this limit,

$$\frac{dy}{dx} = m, \qquad \text{(7)}$$

and m is called the rate at x.

Definition of Instantaneous Rate.

The rate of change of y with respect to x, at x, is the value of dy/dx at x.

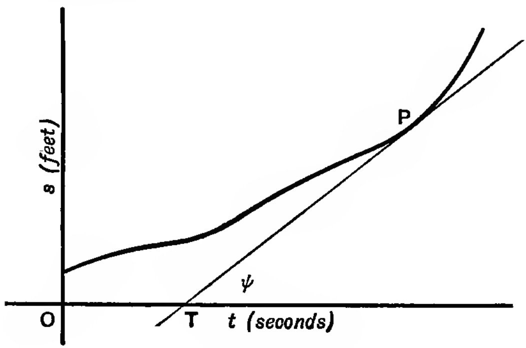


Fig. 9

The study of gradients, or slopes, can now be combined with that of velocity, speed, or rate.

The rate of change of y with respect to x is $\frac{dy}{dx}$, at the selected point.

But $\frac{dy}{dx} = \tan\psi$ where ψ is the angle of slope of the graph of y in terms of x.

$\tan\psi$ is, however, the gradient of the graph at the selected point, hence

the rate of change of y with respect to x, at the selected point, is the same, numerically, as

the gradient of the graph of y in terms of x, at the same point.

When y is distance (s) and x, time (t), the speed or velocity at a selected point is the same, numerically, as the gradient of the distance-time graph at that point.

For instance, the velocity of a body, whose distance-time curve is shown in fig. 9, is $\tan\psi$ at P, where PT is the tangent line at P and ψ the angle of slope.

This result gives a graphical method of finding the velocity of a train, for instance.

Graphical Method of Finding Velocity.

Suppose fig. 10 is the distance-time curve for a suburban train.

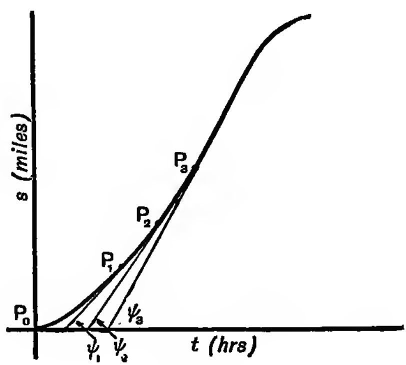


Fig. 10

To deduce the velocity curve from this curve we have to select a range of points, P_0, P_1, P_2, P_3, &c.; draw the tangents at P_0, P_1, P_2, &c., and measure the respective angles of slope ψ_0, ψ_1, ψ_2, &c. Look up the tangents of these angles in a table of tangents, and we get

$$\tan\psi_0,\ \tan\psi_1,\ \tan\psi_2,\ \&c.$$

These numbers are the velocities at the points P_0, P_1, P_2, &c. A curve of velocity against time can now be plotted by plotting these values for the velocity at t_0, t_1, t_2, t_3, &c., which are the times corresponding to P_0, P_1, P_2, &c.

It is convenient, though not essential, to make the intervals of time equal.

Scales.

The velocity is $\tan\psi$, when the scales are unit scales, i.e. 1 unit of distance is represented by 1 in.

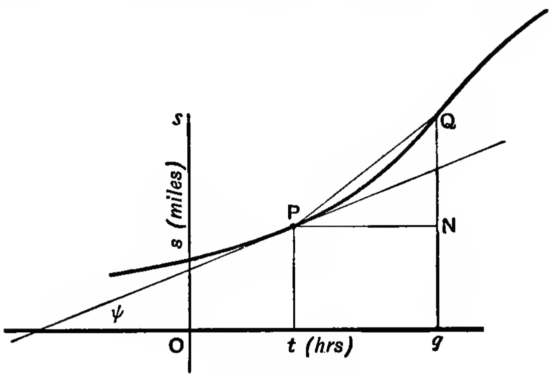


Fig. 11

and 1 unit of time by 1 in. It is often inconvenient to use unit scales, so we take a scale:

$$m \text{ ml.} \equiv 1 \text{ in.}$$
$$n \text{ hr.} \equiv 1 \text{ in.}$$

Let fig. 11 represent a portion of a distance-time curve drawn to this scale.

Let P be a chosen point on the curve, Q a neighbouring point, and PN parallel to Ot.

Then $\tan\psi$, where ψ is the angle of slope as actually drawn, is the limit of $\frac{\text{NQ}}{\text{PN}}$ as Q $\rightarrow$ P.

$$\text{But NQ} = \frac{\Delta s}{m}\text{ in.}$$

$$\text{and PN} = \frac{\Delta t}{n}\text{ in.}$$

$$\therefore \frac{\text{NQ}}{\text{PN}} = \frac{n}{m}\frac{\Delta s}{\Delta t}.$$

$\therefore \tan\psi$ is the limit of $\frac{n}{m}\frac{\Delta s}{\Delta t}$ as Q $\rightarrow$ P, i.e. as $\Delta t \rightarrow 0$.

$$\therefore \tan\psi = \frac{n}{m}\frac{ds}{dt} = \frac{n}{m}v.$$

$$\therefore v\text{, at P,} = \left[\frac{m}{n}\right]\tan\psi, \text{ ....(8)}$$

where ψ is the angle of slope of the graph as actually drawn, and $\left[\frac{m}{n}\right]$ the factor to correct for the scales actually used. v is, of course, measured in the unit corresponding to the units of distance and time used.

This method is not a very good one, in practice, as it is difficult to draw the tangent lines accurately. Later on we shall explain a better method, which depends on having the speed-time curve given instead of the distance-time curve (p. 109).

We have now obtained two ideas of the nature of the limit denoted by dy/dx. They are:

1. It is the measure of the gradient of the graph of y in terms of x, at the point (x, y).

2. It is the measure of the rate of change of y with respect to x, at x.

These two ideas are essentially the same, and are of great importance in the applications of the calculus.

Exercise 2

1. Write down the equation of the straight line joining the points (2, 3) and (−5, 6). What is its gradient?

2. Find the equation of a straight line through the point (3, 7), and making an angle of 30° with OX.

3. Find the equation of the straight line through the point (− 5, 3), which is parallel to the straight line $y = 2x + 7$.

4. What is the gradient of the line

$$3x + 4y - 7 = 0?$$

5. Determine the value of dy/dx at the point (2, 4) on the parabola $x^2 = y$. Write down the equation of the tangent to the parabola at that point.

6. The velocity of a train is varying from time to time. What is the symbol for the rate at which the *velocity* changes with time, at any particular instant t? This rate is called the *acceleration*.

7. £1 is invested at compound interest at r% per annum. Show that the amount (i.e. the original sum plus accumulated interest) is given by

$$M = R^n$$

$$\text{where } M = \text{amount}$$

$$\text{and } R = 1 + \frac{r}{100}.$$

Hence show that if the money is undisturbed for 10 years, the amount from a 6% investment will exceed that from a 5% investment by approximately 3 shillings.

8. The side of a cube is found by measurement to be 9 in. If there is an error of $\frac{1}{100}$ in. in this measurement,

find an approximate value for the consequent error in the calculated volume.

9. Two men, A and B, start from a place O to walk in perpendicular directions, A at 3 miles per hour, and B at $4\frac{1}{2}$ miles per hour. At what rate are they receding from each other.

10. A cube of copper 10 in. long is heated from 0° C. to 10° C. Find approximately the increase in volume. Take the coefficient of linear expansion for copper as 0·000017.

CHAPTER III

Integration

"A snapper-up of unconsidered trifles."—*The Winter's Tale.*

Areas and Integrals.

Let RS be a portion of the graph of $y = f(x)$ between $x = r$ and $x = s$, fig. 1 (i). Let the area RrsS be required, where rR and sS are parallel to OY and rs is the intercept on the OX axis.

This area can be found in many ways. For instance, it can be found by a planimeter, which is an instrument for measuring the areas of plane figures.

Suppose the area OpPH, where P is any point on the graph and pP the ordinate of P, is denoted by A. Let the ordinate r'R′, fig. 1 (ii), represent the area A_r, i.e. OrRH, corresponding to $x = r$; and s'S′, the area A_s, i.e. OsSH, corresponding to $x = s$, and so on. Then p'P′, fig. 1 (ii) represents the area OpPH, A or A_x, corresponding to Op $= x$. A is a function of x, for to each value of x there corresponds a definite value of A.

$$\therefore \text{A} = \text{area O}p\text{PH} = \psi(x), \text{ say.}$$

Now, if we fix attention on P, the increase in A as p advances a little to the right must be smaller and smaller, as the advance in p is made smaller and smaller; in fact, there must be a *gradual* growth in A as p advances to the right. We should therefore expect the "area" graph (A, x) to be a smooth curve between R′ and S′, i.e. to represent a continuous function of x.

Suppose y increases steadily from rR to sS, fig. 1 (i), then the area $pqQP$, which is the increase in area A when x increases from Op to Oq, lies between the

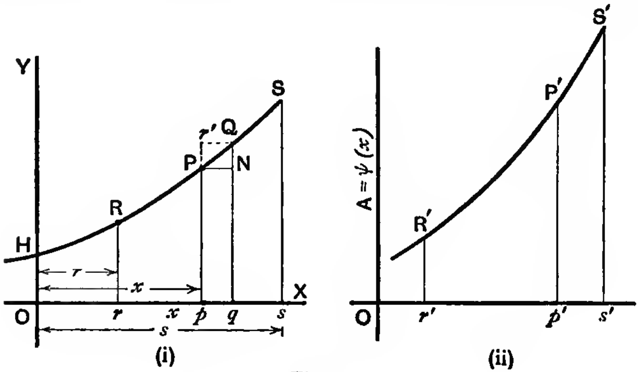


Fig. 1

areas of the rectangles $pqNP$ and $pqQr'$, i.e. the areas are in the following order of magnitude:

$$pqNP < pqQP < pqQr',$$

$$\text{i.e. } pP \cdot pq < pqQP < qQ \cdot pq,$$

$$\text{i.e. } y\Delta x < \Delta A < (y + \Delta y)\Delta x,$$

$$\text{i.e. } y < \frac{\Delta A}{\Delta x} < y + \Delta y.$$

Since the curve RS is continuous, as $\Delta x \to 0$, Δy also $\to 0$, and since $\frac{\Delta A}{\Delta x}$ always lies in value between y and $y + \Delta y$,

$$\underset{\Delta x \to 0}{\text{Lt}} \frac{\Delta A}{\Delta x} = y, \text{ i.e.}$$

$$\frac{dA}{dx} = y. \quad \text{.....................(1)}$$

The derivative of the "area" function, fig. 1 (ii), is therefore the ordinate of the (x, y) curve, fig. 1 (i), at any value of x, between $x = r$ and $x = s$.

The "area" function, $\psi(x)$, is therefore a function whose *derivative*, at any point, is $f(x)$, the *given* function.

$\psi(x)$, therefore, satisfies the equation

$$\frac{d}{dx}\psi(x) = f(x),$$

where $f(x)$ is the given function.

In the differential calculus we are *given* $\psi(x)$ and asked to find $f(x)$. Here we are *given* $f(x)$ and asked to find $\psi(x)$. Hence we have to work the processes of the differential calculus backwards in order to find areas, for which purpose a *special notation* is convenient.

If $\psi(x)$ is given, we say that

$$\frac{d}{dx}\psi(x) = f(x), \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

where $\frac{d}{dx}$ denotes the operation of finding $f(x)$ from $\psi(x)$ by the rules of the differential calculus.

In the present case we write

$$\psi(x) = \int f(x)dx, \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

which is exactly the same statement, in another notation, as equation (2).

Equation (3) is read

$$\psi(x) = \text{the } \textit{integral} \text{ of } f(x);$$

$$\text{equation (2), } f(x) = \text{the } \textit{derivative} \text{ of } \psi(x).$$

The reader will notice that, if

$$\frac{d}{dx}\psi(x) = f(x),$$

$$\text{then } \psi(x) = \int f(x)dx;$$

or putting z for $\psi(x)$, if

$$\frac{dz}{dx} = f(x) = \text{X, say,} \quad \ldots\ldots(4)$$

$$\text{then } z = \int \text{X}dx. \quad \ldots\ldots\ldots\ldots\ldots(5)$$

Equations (4) and (5) are not *deductions*, one from the other, but merely different notations for expressing the same fact, namely, *the derivative of z is X*, for any value of x at which (4) and (5) hold. If we are dealing with a problem of the differential calculus, notation (4) is convenient; if the problem is one of the integral calculus, notation (5) is convenient. The reason for using the sign $\int$ has been mentioned on p. 16.

EXAMPLE

If $\psi(x) = x^n$, where n is a positive integer, then

$$f(x) = nx^{n-1},$$

because

$$\frac{d}{dx}x^n = nx^{n-1}$$

by Rule 1, p. 44, and, in the new notation,

$$x^n = \int nx^{n-1}dx.$$

Here x^n is the "area" function corresponding to the given function, nx^{n-1}.

$\psi(x)$ is not a fixed quantity, but a function of x. The area under RS *is*, however, a fixed quantity, 10 sq. in., say.

If we know $\psi(x)$, we can plot $A = \psi(x)$, fig. 1 (ii), therefore this curve becomes known when $\psi(x)$ is known.

$$\text{At } x = r, \ \psi(x) = \psi(r).$$
$$\text{At } x = s, \ \psi(x) = \psi(s).$$

But the area RrsS required is the area A up to $x = s$, less the area up to $x = r$; no matter what value of x (less than s) we start measuring the area from.

The area required is $(A_s - A_r)$, i.e.

$$\psi(s) - \psi(r).$$

When the integral is to be taken between definite boundary values of x in this way, we write

$$\int_r^s f(x)dx = \psi(s) - \psi(r), \ldots\ldots\ldots\ldots(6)$$

where r is called the lower limit, and s the upper limit, and $\psi(x) = \int f(x)dx$.

Such an integral is no longer a function of x. It is a definite quantity, $\psi(s) - \psi(r)$, and it is called *a definite integral.* When the limits are not specified, the integral *is a function of x*, and is *not* a definite quantity. It is called an *indefinite* integral. The process of finding $\psi(x)$, given $f(x)$, i.e. of finding $\int f(x)dx$, is called *integration.*

Evaluation of Definite Integrals.

The value of the definite integral

$$\int_a^b f(x)dx$$

can be found in at least three ways.

1. By equation (6), the integral represents the area of the figure AabB (fig. 3), and this area can be

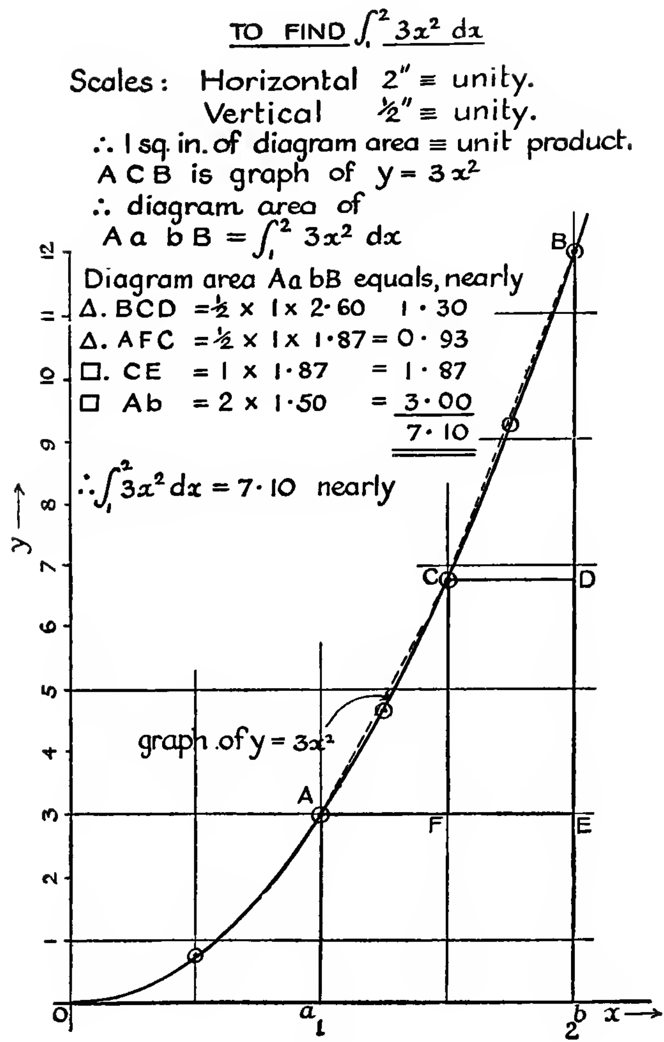

TO FIND $\int_1^2 3x^2\,dx$
Scales: Horizontal 2" ≡ unity.
Vertical ½" ≡ unity.
∴ 1 sq. in. of diagram area ≡ unit product.
A C B is graph of $y = 3x^2$
∴ diagram area of
A a b B = $\int_1^2 3x^2\,dx$
Diagram area Aa bB equals, nearly
Δ. BCD = ½ × 1 × 2·60 1·30
Δ. AFC = ½ × 1 × 1·87 = 0·93
□. CE = 1 × 1·87 = 1·87
□ Ab = 2 × 1·50 = 3·00
7·10
∴ $\int_1^2 3x^2\,dx$ = 7·10 nearly
graph of y = 3x²
B
C
D
A
F
E
O
a
1
b
2
x →
y →
1
2
3
4
5
6
7
8
9
10
11
12

Fig. 2

readily found when the graph of $y = f(x)$ is drawn. This method has already been touched upon in the Introduction.

As an example, find graphically

$$\int_1^2 3x^2 dx.$$

The graph of $y = 3x^2$ must first be plotted.

Take 2 in. horizontally equal to 1, and $\frac{1}{2}$ in. vertically equal to 1. ∴ 1 sq. in. of area in the diagram represents one unit of real area.

The figure is drawn in fig. 2.

Full details of the calculation are given on the figure.

It will be seen that the result is 7·10.

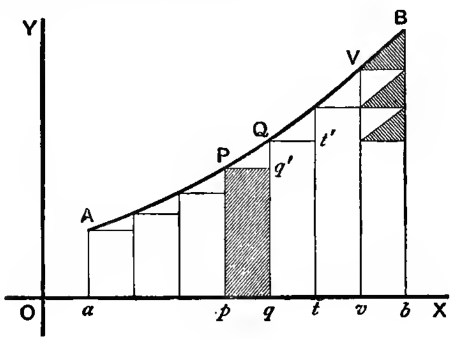


Fig. 3

2. The name "integration" suggests adding up, just as differentiation suggests taking differences. We have already seen in the Introduction that there is an addition buried in the preceding work.

Divide ab (fig. 3) into any number n of equal parts, and erect ordinates at the points of division. Let

$$Op = x_p, \; Oq = x_q, \; \&c.$$

Find the area of each elementary rectangle, such as $Ppqq'$. The area of this rectangle is

$$pP \times pq = f(x_p) \times \Delta x.$$

The area of

$$Qqtt' = f(x_q) \times \Delta x,$$

and so on for each area. Hence the sum of all the rectangles between $x = Oa$ and $x = Ob$ is

$$\Sigma_a^b\{f(x)\Delta x\} = \text{area required } nearly. \ldots\ldots(7)$$

Consider the word "nearly" in (7) a little further.

The neglected areas are $\Sigma(Pq'Q)$. Translate each of these areas parallel to OX into the final area $VvbB$. The sum of them *all* is manifestly less than the area $VvbB$.

But this area itself is less than $f(b) \times vb$.

∴ the sum of the neglected areas is less than $f(b) \times vb$.

Now, $f(b)$ is finite, hence

$$f(b) \times vb \rightarrow 0 \text{ as } vb \rightarrow 0,$$
$$\text{i.e. } f(b) \times \Delta x \rightarrow 0 \text{ as } \Delta x \rightarrow 0,$$

since all the subdivisions pq, qt, . . . vb are equal by construction.

Hence

$$\Sigma_a^b f(x)\Delta x \rightarrow \text{the area } AabB \text{ as } \Delta x \rightarrow 0,$$

$$\text{i.e. } \underset{\Delta x \rightarrow 0}{\text{Lt}} \Sigma_a^b f(x)\Delta x = \text{area } AabB$$

$$= \int_a^b f(x)dx. \ldots\ldots\ldots\ldots\ldots\ldots(8)$$

To determine the integral, i.e. the area in question, to any required degree of accuracy, it is necessary to divide ab into parts sufficiently small to ensure that the ratio of $[f(b) \times \Delta x]$ to $\Sigma_a^b\{f(x)\Delta x\}$ is less than the given fraction which defines the degree of accuracy.

This example brings out the fact that integration is really addition, more or less disguised, and immediately suggests our finding the value of

$$\int_a^b f(x)dx$$

by direct addition, as this integral is

$$\underset{\Delta x \to 0}{\text{Lt}} \ \Sigma_a^b f(x)\Delta x. \quad \text{..................(9)}$$

The finding of this limit is simply a matter of arithmetic, using progressively (and sufficiently) diminishing values for Δx, depending on the accuracy required in the result.

We can write down a list of all the products $f(x)\Delta x$, taking $\Delta x = \frac{1}{100}(b-a)$, say.

At $x = a, f(x) = f(a)$; the first product is therefore

$$f(a)\frac{(b-a)}{100} = P_1\frac{(b-a)}{100}.$$

The length of the next strip is the value of $f(x)$ at

$$x = \left[a + \frac{(b-a)}{100}\right] \text{ i.e. } f\left\{a + \frac{(b-a)}{100}\right\} = P_2.$$

The area of the second strip is therefore

$$P_2\frac{(b-a)}{100}.$$

There will be, in all, 100 strips. The sum of them all is

$$[P_1 + P_2 + P_3 + \ldots + P_{100}]\left(\frac{b-a}{100}\right)$$

$$= \Sigma_1^{100} P_n\left(\frac{b-a}{100}\right). \quad (10)$$

This value is the sum

$$\Sigma_a^b f(x)\Delta x, \text{ when } \Delta x = \left(\frac{b-a}{100}\right).$$

As we make Δx a smaller and smaller fraction of $(b-a)$, the value of

$$\Sigma_a^b f(x)\Delta x$$

will approach nearer and nearer to the value we want, viz.

$$\int_a^b f(x)dx,$$

which is *by definition* the limit to which

$$\Sigma_a^b f(x)\Delta x \text{ tends as } \Delta x \to 0.$$

The arithmetic is evidently lengthy if high accuracy is aimed at. We will take a worked example with fairly large intervals, and compare our result with accurate results which we shall obtain later.

WORKED EXAMPLE

Find an approximate value of $\int_1^2 3x^2dx$.

We have

$$\int_1^2 3x^2dx \approx \Sigma_1^2 3x^2\Delta x.$$

Take

$$\Delta x = \frac{b-a}{10} = \frac{1}{10}(2-1) = \frac{1}{10}.$$

Then $f(1) = 3$, hence $3 \times \frac{1}{10} = 0{\cdot}300$ is the first term of the sum.

The calculation can be best pursued in a tabular form. Notice that the factor 3 goes right through the arithmetic as a factor, hence

$$\Sigma_1^2 3x^2\Delta x = 3\Sigma_1^2 x^2\Delta x.$$

We shall therefore take $\Sigma_1^2 x^2\Delta x$, and multiply the answer by 3.

Interval.	x^2 at Beginning of Interval.	P_n.	$P_n\Delta x$.
1 –1·1	1·00	1·00	0·100
1·1–1·2	1·21	1·21	0·121
1·2–1·3	1·44	1·44	0·144
1·3–1·4	1·69	1·69	0·169
1·4–1·5	1·96	1·96	0·196
1·5–1·6	2·25	2·25	0·225
1·6–1·7	2·56	2·56	0·256
1·7–1·8	2·89	2·89	0·289
1·8–1·9	3·24	3·24	0·324
1·9–2·0	3·61	3·61	0·361

$$\Sigma P_n\Delta x = 2 \cdot 185.$$

$$\therefore \quad \Sigma_1^2 x^2\Delta x = 2 \cdot 185.$$

$$\therefore \quad \Sigma_1^2 3x^2\Delta x = 6 \cdot 555.$$

If, in column 2, we take x^2 at the *end* of each interval, and go through the same calculations, we get 7·455. The mean of these is 7·005, and this, we shall see, is very close to the true value.

3. Yet a third method is available.

We have shown that, if

$$\psi(x) = \int f(x)dx,$$

$$\text{then } f(x) = \frac{d}{dx}\psi(x).$$

$$\therefore \quad \psi(x) = \int \frac{d}{dx}\psi(x)\,dx.$$

Putting $y = \psi(x)$, we get

$$y = \int\left(\frac{dy}{dx}\right)dx. \qquad \ldots\ldots\ldots\ldots\ldots(11)$$

If $y = x^n$, n being a positive integer,

$$\frac{dy}{dx} = nx^{n-1} \text{ by Rule 1.}$$

$$\therefore x^n = \int nx^{n-1}dx. \qquad \text{.....................(12)}$$

We have thus found the value of the integral $\int nx^{n-1}dx$ *in terms of* x.

Suppose we wish to find

$$\int_1^2 nx^{n-1}dx.$$

We have

$$x^n = \int nx^{n-1}dx = \text{our } \psi(x) \text{ of pp. 71 to 75.}$$

$\therefore$ the value of the integral between $x = 1$ and $x = 2$ is $(2^n - 1^n)$, by equation (6),

$$\text{i.e. } 2^n - 1^n = \int_1^2 nx^{n-1}dx.$$

Generally,

$$b^n - a^n = \int_a^b nx^{n-1}dx \qquad \text{............(13)}$$

It will be seen that this method consists in working backwards from known rules of the differential calculus. It is in this sense that the integral calculus is the inverse of the differential calculus.

Rule II.

$$\int nx^{n-1}dx = x^n,$$

i.e. the integral of n times x^{n-1} is x^n, *when n is a positive integer (excluding $n = 0$).*

The finding of the integral as a function of x is a great advance on the other two methods. Either of these two methods takes time and is approximate;

but the third method is *accurate* and quickly applied, e.g. we can quickly check the result for $\int_1^2 3x^2dx$.

$$\int_1^2 3x^2dx = \left[x^3\right]_1^2 = 2^3 - 1^3 = 7.$$

The value obtained by arithmetic was 7·005, and by the graphical method 7·1.

Exercise 3

1. Find by the arithmetical method, taking ten steps,

$$\int_1^5 5x^4dx.$$

2. Find by the graphical method,

$$\int_1^5 5x^4dx.$$

3. Find by Rule II,

$$\int_1^5 5x^4dx.$$

Compare the three results.

4. Find by the arithmetical method, taking ten steps,

$$\int_3^5 e^x dx,$$

taking your values of the function e^x from a book of tables. Is it approximately equal to $(e^5 - e^3)$? What does this result suggest? [Take $e = 2{\cdot}718$ in this example.]

5. Find by the graphical method,

$$\int_0^\pi \sin x dx.$$

Is it approximately equal to $(\cos 0^\circ - \cos \pi)$? What does this suggest? (x is in radians, see p. 102.)

6. Find by the graphical method,

$$\int_0^1 x^2(1 - x)^2dx.$$

7. It was found, p. 56, that

$$\frac{ds}{dt} = v,$$

where v is the velocity at t, and s is the distance travelled by a body moving in a straight path from rest in an interval of time t. Express the distance travelled as an integral of velocity and time. Assuming the velocity is proportional to the square of the time t, and the constant of proportionality to be 3μ ($v = 3\mu t^2$), if distances are measured in feet and times in seconds, show that the distance travelled from rest in T sec. is μT^3 ft. Express this result in words.

CHAPTER IV

Limits

"Long calculations or complex diagrams affright the timorous and unexperienced from a second view; but if we have skill sufficient to analyse them into simple principles, it will be discovered that our fear was groundless."—SAMUEL JOHNSON.

We have already used the word "limit" to denote a quantity to which another quantity can be made to approach as nearly as we please.

We will now examine the nature of a "limit" more closely, as this idea is the very pivot of our subject, and of all higher mathematical analysis. No real progress can be made till the idea is thoroughly grasped.

As a simple case, let us seek the limit of x^2 at $x = 10$ (fig. 1).

At the outset, a warning is necessary. *The limit has nothing to do with the value of the function at* $x = 10$. It depends on the values of the function *round about* $x = 10$, but not *at* $x = 10$. We deliberately exclude the value $x = 10$ and the corresponding value of the function in finding the limit required. The reader will soon see why this is so.

We have already noticed that, if there is a limit, it is *approached* as x approaches the given value.

Suppose

$x = 9 \cdot 9$, then $x^2 = 98 \cdot 01$
$x = 9 \cdot 99$, ,, $x^2 = 99 \cdot 8001$
$x = 9 \cdot 999$, ,, $x^2 = 99 \cdot 980001$.

From these figures, it appears that

$x^2 \to 100$ as $x \to 10$ from values of x less than 10.

Now, try some values of x greater than 10.

Suppose

$$x = 10{\cdot}1, \quad \text{then } x^2 = 102{\cdot}01$$
$$x = 10{\cdot}01, \quad ,, \quad x^2 = 100{\cdot}2001$$
$$x = 10{\cdot}001, \quad ,, \quad x^2 = 100{\cdot}020001.$$

These values of x^2 again approach 100 as $x \to 10$.

The figures suggest that 100 is the limit we are seeking. Let us test this suggestion.

Tabulating these figures,

x	x^2	$\lvert x^2 - 100 \rvert$	$\lvert x - 10 \rvert$
9·9	98·01	1·99	0·1
9·99	99·8001	0·1999	0·01
9·999	99·980001	0·019999	0·001
10·001	100·020001	0·020001	0·001
10·01	100·2001	0·2001	0·01
10·1	102·01	2·01	0·1

we see that:

(1) So long as x lies between 9·9 and 10·1 (excluding $x = 10$), the difference between x^2 and 100 does not exceed 2·01 numerically, i.e. provided x is not equal to 10,

if $9{\cdot}9 < x < 10{\cdot}1$, then $0 < \lvert x^2 - 100 \rvert < 2{\cdot}01$.

(2)

If $9{\cdot}99 < x < 10{\cdot}01$, ,, $0 < \lvert x^2 - 100 \rvert < 0{\cdot}2001$.

(3)

If $9{\cdot}999 < x < 10{\cdot}001$, ,, $0 < \lvert x^2 - 100 \rvert < 0{\cdot}020001$.

Each of these intervals of x contains the prohibited value $x = 10$, and as the interval shrinks, the bounds for $| x^2 - 100 |$ shrink too — the third and fourth columns in the table show this. It therefore seems likely that, no matter how small a quantity we select for the upper bound of $| x^2 - 100 |$, we can find an interval containing the value 10 such that if x lies in this interval, and does not have the value 10, the

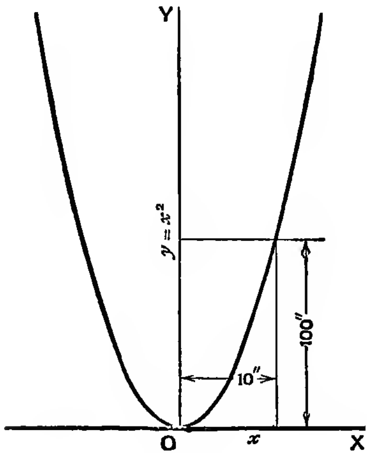


Fig. 1

value of $| x^2 - 100 |$ lies between this selected upper bound and zero.

We will show that this is so. By looking at the graph of x^2 (fig. 1) it is obvious that x^2 increases as x increases and x^2 decreases as x decreases, provided x is always positive.

Select a number $\frac{1}{10^{12}}$, which would probably denote a small quantity if we were using it with a usual unit, and see whether we can choose an interval so that if x lies in this interval, then

$$0 < |x^2 - 100| < \frac{1}{10^{12}}.$$

Case I. $x > 10$.

Suppose first that x is greater than 10.

So long as

$$10 < x < \sqrt{100 + \frac{1}{10^{12}}},$$

$$\text{then } 100 < x^2 < 100 + \frac{1}{10^{12}},$$

$$\text{i.e. } 0 < x^2 - 100 < \frac{1}{10^{12}},$$

$$\text{i.e. } (x^2 - 100) \text{ lies between } 0 \text{ and } \frac{1}{10^{12}}.$$

If then x lies in the interval *between* $x = 10$ and

$$x = +\sqrt{100 + \frac{1}{10^{12}}}$$

(and does not have either of these "end values"), then the inequality

$$0 < x^2 - 100 < \frac{1}{10^{12}}$$

holds good.

Now, suppose, instead of $\frac{1}{10^{12}}$, we use $\frac{1}{10^{12000}}$, i.e. 0·0000...... 1, with 11,999 noughts between the decimal point and the 1 at the end. We can still find an interval, viz. from $x = 10$ to

$$x = +\sqrt{100 + \frac{1}{10^{12000}}},$$

such that, if x lies in it, the inequality

$$0 < x^2 - 100 < \frac{1}{10^{12000}}$$

holds good.

We will now try to put *any* number, *excluding zero*, for the fraction. For definiteness, we will first suppose the number to be $< \frac{1}{10^{12}}$. If ϵ be such a number, we easily see that so long as x lies in the interval between

$$x = 10 \text{ and } x = +\sqrt{100 + \epsilon},$$

then $0 < | x^2 - 100 | < \epsilon$ holds.

The difference between x^2 and 100 can therefore be made less than *any* assignable number ϵ $\left(\text{less than } \frac{1}{10^{12}}\right)$, *excluding zero*, by choosing a suitable interval within which x must lie.

Case II. $x < 10$.

We have so far supposed $x > 10$.

Now, suppose $x < 10$; x^2 is then less than 100, and the positive difference between x^2 and 100 is

$$100 - x^2.$$

So long as the inequality

$$10 > x > +\sqrt{100 - \frac{1}{10^{12}}} \text{ holds,}$$

then $100 > x^2 > 100 - \frac{1}{10^{12}}$, by squaring each term,

$$\text{i.e. } 0 > x^2 - 100 > -\frac{1}{10^{12}}$$

$$\text{i.e. } 0 > -(100 - x^2) > -\frac{1}{10^{12}}$$

Now $(100 - x^2)$ is positive, therefore $-(100 - x^2)$ and $-\frac{1}{10^{12}}$ are negative numbers.

If a given inequality runs in ascending (or descending) order of magnitude, it runs in the reverse order

—descending (or ascending) when the signs of all the terms are changed. For instance

$$3 > 2 > 1,$$
$$\text{but } -3 < -2 < -1.$$
$$\text{If then } 0 > -(100 - x^2) > -\frac{1}{10^{12}},$$
$$\text{then } 0 < 100 - x^2 < \frac{1}{10^{12}}.$$

Hence so long as x lies in the interval

$$\left[10, \ +\sqrt{100 - \frac{1}{10^{12}}}\right],$$

$(100 - x^2)$ or $|x^2 - 100|$ lies in the interval

$$\left[0, \ \frac{1}{10^{12}}\right].$$

This argument is quite general, and if we put ϵ for *any* positive number $\left(\text{less than } \frac{1}{10^{12}}\right)$, we get

$$0 < (100 - x^2) < \epsilon$$

so long as x lies between $+\sqrt{100 - \epsilon}$ and 10, where ϵ can be *any* positive number which is less than $\frac{1}{10^{12}}$. We have now proved that

1. If $x > 10$, $0 < |x^2 - 100| < \epsilon$,

provided x lies between 10 and $+\sqrt{100 + \epsilon}$, where ϵ can be any finite positive number $< \frac{1}{10^{12}}$.

2. If $x < 10$, then $0 < |x^2 - 100| < \epsilon$,

provided x lies between 10 and $+\sqrt{100 - \epsilon}$, where ϵ can be any finite positive number $< \frac{1}{10^{12}}$.

There is no need to examine specially values of $\epsilon > \frac{1}{10^{12}}$, for if $\epsilon > \frac{1}{10^{12}}$, we only have to keep x between

$$+\sqrt{100 - \frac{1}{10^{12}}} \text{ and } +\sqrt{100 + \frac{1}{10^{12}}},$$

excluding $x = 10$, to ensure that

$$0 < | x^2 - 100 | < \epsilon,$$

for this limitation of x is sufficient to ensure that the inequality

$$0 < | x^2 - 100 | < \frac{1}{10^{12}} \text{ holds, } \ldots\ldots\ldots(1)$$

and if $\epsilon > \frac{1}{10^{12}}$ the inequality $0 < | x^2 - 100 | < \epsilon$ must necessarily hold, for the same interval of x.

We can therefore always find an interval including $x = 10$, *such that if x lies within this interval, the inequality*

$$0 < | x^2 - 100 | < \epsilon \text{ holds, } \ldots\ldots\ldots(2)$$

except at the value $x = 10$, *no matter what finite positive value we give to* ϵ, *except zero.*

When condition (2) holds, we say, as on p. 39, that $| x^2 - 100 |$ tends to zero as x tends to 10, i.e. x^2 tends to *the limit* 100.

It is clear that if we choose progressively decreasing values of ϵ, the permissible range of x becomes more and more restricted, for the range D is given by

$$D = \sqrt{100 + \epsilon} - \sqrt{100 - \epsilon},$$

$$\text{i.e. } D^2 = 200 - 2\sqrt{100^2 - \epsilon^2},$$

and if ϵ is given progressively decreasing values,

$$\frac{1}{10^{12}}, \quad \frac{1}{10^{12000}}, \quad \frac{1}{10^{12000000}}$$

and so on, $(100^2 - \epsilon^2)$ takes progressively increasing values, and therefore $[200 - 2\sqrt{100^2 - \epsilon^2}]$ progressively decreasing values, hence D progressively decreases.

The values of x therefore become restricted to those "in the neighbourhood" of 10, 10 being *always* included in the interval.

We therefore say that

$$| x^2 - 100 | \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

tends to zero *in the neighbourhood of* 10, i.e. x^2 tends to *the limit* 100 in the neighbourhood of $x = 10$.

As the admissible range of x shrinks more and more, the only value of x we can be sure of including in the range is 10, just as the only point which certainly remains inside a circle, if we shrink the radius more and more, is the centre of the circle. Now this progressive shrinking of the interval is not only admissible, but is the very essence of the process. The value $x = 10$ is the nucleus, so to speak, of the shrinking interval, and as it is the only point we are *always* sure of securing in the interval, it is reasonable to associate the limit we have found with this value of x. We therefore say that the limit of x^2 *at* $x = 10$ is 100.

If we are dealing with quantities we can put the matter thus:—

The quantity x^2 *can be made to differ from* 100 *by as small a quantity as we please, by bringing* x *sufficiently near to* 10, *no matter whether it is more or less than* 10.

In such circumstances we say that the quantity x^2 has the limit 100 *at* $x = 10$.

The reader will now see that the limit of x^2 is a number we associate with a given value of x, here $x = 10$, and this number is obtained by considering the *values* of x^2 corresponding to values of x *round about* 10 as these values of x progressively approach 10; it is not dependent in any way on the *value* of x^2 at $x = 10$, a value which has been deliberately excluded from consideration all through the calculations.

Definition of a Limit.

$f(x)$ has the limit l at $x = a$, if, corresponding to any finite positive quantity ϵ, however small we like to make it, we can find an interval of x, including the value $x = a$, such that the inequality

$$0 < | f(x) - l | < \epsilon \text{ holds,}$$

so long as the inequality

$$0 < | x - a | < \eta \text{ holds,}$$

where η is a finite positive quantity to be found, which depends on the value chosen for ϵ.

As ϵ decreases η decreases.

In other words, the limit l exists if

$$| f(x) - l | \text{ tends to zero}$$

as x tends to a.

Or we may put it still more briefly, in symbols.

$$\text{If, as} \quad x \rightarrow a,$$
$$f(x) \rightarrow l,$$

then l is the limit of $f(x)$ at $x = a$.

This shorter statement is, however, only apparently shorter, as it implies the meaning of $\rightarrow$ which is expressed in words in the longer statement. It is, however, a very useful pictorial notation, as the sign $\rightarrow$ suggests vividly the process of finding the limit.

It should be noticed that the statement in the definition excludes the value $x = a$; for the conditional inequality is $0 < | x - a | < \eta$, i.e. $| x - a |$ must not equal zero, i.e. x must not equal a. Further, it implies nothing whatever about the value of $f(x)$ at $x = a$.

This example has been worked out at length in order to bring out the full details of the idea of a "limit". Under ordinary circumstances, we should not go through the whole of the working, as it is obvious that

$$x^2 \longrightarrow 100$$
$$\text{as } x \longrightarrow 10.$$

The point is that the full working out expresses quite definitely, by an arithmetical method based on inequalities, what appears to be obvious from the graph, and, as the idea of the limit is so important, it is worth while to consider one case at least in detail. Further, limits are not always obvious, and *sometimes a graph cannot be drawn.* Skill in the use of inequalities is very useful in all branches of mathematics and physics. By using inequalities in a common-sense way we can often set definite limits to approximate calculations, and applied mathematics, physics, and engineering all bristle with these. For example, a glance at fig. 2, p. 76, shows that the area of the figure is greater than $f(a) \times (b - a)$, and less than $f(b) \times (b - a)$. Hence

$$f(a) \times (b - a) < \int_a^b f(x)dx < f(b) \times (b - a),$$

if $f(x)$ is positive and continually increases as x increases from a to b; and, whether we can integrate

$$\int_a^b f(x)dx$$

or not, we undoubtedly can work out $f(a) \times (b - a)$ and $f(b) \times (b-a)$, and thus get bounds to the integral within which it must lie. These bounds are often quite close enough for practical purposes, so that we need not integrate the expression.

Values of Functions—Continuous Functions.

We have found that the value of the function x^2 at $x = 10$ has no bearing on the *limit* of x^2 at $x = 10$. The limit at $x = 10$ is 100 and the value of x^2 at $x = 10$ is 100. The limit equals the value of the function in this case. As a matter of fact, nearly all the ordinary functions we come across in applied mathematics possess the property that the value of the function, for a given value of the variable x, is identical with the limit of the function at that value of x.

Such functions are called *continuous* functions.

Suppose $f(x)$ has the limit l at $x = a$, then if $f(x)$ is continuous at

$$x = a,$$
$$f(a) = l.$$

It is evident, then, that when $f(x)$ is continuous at $x = a$

$$f(x) \to f(a) \text{ as } x \to a.$$

This is manifestly the state of things at any point on a smooth graph.

If $y = f(x)$ is continuous at *every point* between a and b, then if y and x are the co-ordinates of any point on the graph of $f(x)$ between a and b, then

$$\Delta y \to 0 \text{ as } \Delta x \to 0.$$

This follows from the fact that

$$f(x + \Delta x) \to f(x) \text{ as } x + \Delta x \to x,$$
$$\text{i.e. } y + \Delta y \to y \text{ as } x + \Delta x \to x,$$
$$\text{i.e. } \Delta y \to 0 \text{ as } \Delta x \to 0.$$

This property of continuous functions is most important.

Undetermined Functions.

In the calculus, certain important functions crop up which have *no value* for certain values of x, and it is just at these values of x that we specially want to know something about the behaviour of the function. Fortunately, it is not necessary for us to know the value of the function *at* these special values of x; what we want to know is something about the values of the function for values of x *near* these special values of x. The limit gives this information concisely. As we have already seen, we do not need to know the value of the function at a selected value of x to find the *limit* of the function at that value of x.

WORKED EXAMPLE

Consider the function $\frac{x^2 - 1}{x - 1}$ at $x = 1$.

This function has no value at $x = 1$, for if we put $x = 1$, we get

$$f(1) = \frac{1 - 1}{1 - 1} = \frac{0}{0}.$$

This result is quite useless, as we can attach no meaning to it.

But we can easily show that $\frac{x^2 - 1}{x - 1}$ has a limit at $x = 1$, for $\frac{x^2 - 1}{x - 1}$ lies round about the value 2 when x is near unity. When

$$x = 1{\cdot}01, \quad \frac{x^2 - 1}{x - 1} = \frac{1{\cdot}01^2 - 1}{{\cdot}01} = \frac{1{\cdot}0201 - 1}{{\cdot}01} = 2{\cdot}01.$$

$$x = {\cdot}99, \quad \frac{x^2 - 1}{x - 1} = \frac{0{\cdot}9801 - 1}{0{\cdot}99 - 1} = \frac{-0{\cdot}0199}{-0{\cdot}01} = 1{\cdot}99.$$

These figures point to 2 as the limit, and the reader will have no difficulty in showing, just as we have done for x^2, that we can always find a positive value for η such that

$$0 < \left| \frac{x^2 - 1}{x - 1} - 2 \right| < \epsilon,$$

so long as

$$0 < |x - 1| < \eta,$$

no matter what arbitrary finite positive value we give to ϵ.

We can, however, see that the limit is 2 at $x = 1$, without going into the detail of the inequalities.

$$\frac{x^2 - 1}{x - 1} = x + 1, \text{ when } x - 1 \text{ is not zero,}$$

$$\text{and } \operatorname*{Lt}_{x \to 1} \left(\frac{x^2 - 1}{x - 1} \right) = \operatorname*{Lt}_{x \to 1} (x + 1) = 2.$$

We cannot say that

$$\frac{x^2 - 1}{x - 1} = x + 1 \text{ for all values of } x,$$

And that when $x = 1$, $x + 1 = 2$.

$$\therefore \frac{x^2 - 1}{x - 1} \doteq 2 \text{ when } x = 1,$$

because the first equation is true only when $x - 1$ is not zero, i.e. when x is not *unity*. It is therefore inadmissible to put $x = 1$ in the next line of the argument.

When $x = 1$, division of $x^2 - 1$ by $x - 1$ is *division by zero*, which is not allowed in the laws of algebra (p. 108, Ex. 9), and, worse, it is division of zero by zero, which is an operation without any meaning.

Further, even if the reasoning were valid, it would give the value of $\frac{x^2 - 1}{x - 1}$ at $x = 1$, and this value has nothing to do with the limit at $x = 1$ which we require.

EXAMPLES

1. Show that if $y \rightarrow b$ as $x \rightarrow a$, the limit at $x = a$ of $(a + y)$ is $a + b$.

2. Find $\underset{x=a}{\text{Lt}}\ x^3$.

Important Points about Limits.

1. The limit is not necessarily the value of the function at the value of x under discussion—$x = a$, say. The function, $\frac{x^2 - a^2}{x - a}$, e.g. *has no value* at $x = a$.

2. The limit is a number from which the values of the function, *in the immediate neighbourhood* of $x = a$, differ by as little as we please.

3. By calculating the value of the function for a value of x sufficiently near to $x = a$, a value of the function can be found differing as little as we please from the limit.

4. The notation of limits is

$$\underset{x=a}{\text{Lt}}\ f(x) = l,$$

i.e. the limit of $f(x)$ at $x = a$ is l. This notation can be put

$$f(x) \to l \text{ as } x \to a,$$

which reads $f(x)$ approaches the value l as closely as desired when x approaches the value a sufficiently closely.

We will now discuss certain theorems which will be very useful in later work.

Theorem 1. Limit of a Sum or Difference.

The limit of the sum of a finite number of functions of x is equal to the sum of their limits.

Let

$$f_1(x) \to l_1 \text{ as } x \to a,$$
$$f_2(x) \to l_2 \text{ as } x \to a.$$

When x approaches a, $f_1(x)$ approaches l_1 and $f_2(x)$ approaches l_2.

$\therefore f_1(x) + f_2(x)$ must approach $l_1 + l_2$,

$$\text{i.e. } f_1(x) + f_2(x) \to l_1 + l_2 \text{ as } x \to a.$$

We can put this argument thus:

Let PQ (fig. 2) be the graph of $f_1(x)$.

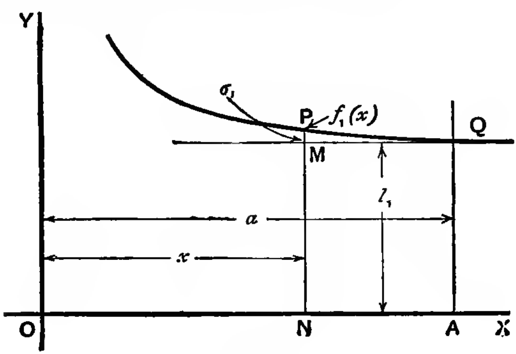


Fig. 2

Then, at any point N given by $x = \text{ON}$,

$$\begin{aligned} \text{NP} &= \text{NM} + \text{MP}, \\ \text{i.e. } f_1(x) &= l_1 + \sigma_1, \\ \text{where } \sigma_1 &= \text{MP}. \end{aligned}$$

In the figure σ_1 and l_1 are positive; in general, they may be either positive or negative quantities.

If $f_1(x)$ has the limit l_1 as $x \to a$,

$$f_1(x) \to l_1 \text{ as } x \to a,$$

i.e. as fig. 2 shows,

$$\sigma_1 \to 0 \text{ as } x \to a.$$

Similarly,

$$f_2(x) = l_2 + \sigma_2, \text{ where } \sigma_2 \to 0 \text{ as } x \to a.$$

Adding

$$f_1(x) + f_2(x) = l_1 + l_2 + \sigma_1 + \sigma_2.$$

Now, as $\sigma_1 \to 0$ and $\sigma_2 \to 0$ when $x \to a$,

$$\therefore \sigma_1 + \sigma_2 \to 0 \text{ when } x \to a.$$

$$\therefore f_1(x) + f_2(x) \to l_1 + l_2 \text{ as } x \to a,$$

$$\text{i.e. } \underset{x=a}{\text{Lt}} [f_1(x) + f_2(x)] = l_1 + l_2.$$

The theorem is thus true for two terms, and can similarly be extended to any number of terms.

Theorem 2. Limit of a Product.

The limit of the product of a finite number of functions of x is equal to the product of their limits, if these limits are all finite.

Let

$$\begin{aligned} f_1(x) &\to l_1 \text{ as } x \to a, \\ f_2(x) &\to l_2 \text{ as } x \to a. \end{aligned}$$

As before, we may put

$$f_1(x) = l_1 + \sigma_1, \text{ where } \sigma_1 \to 0 \text{ as } x \to a,$$
$$\text{and } f_2(x) = l_2 + \sigma_2, \text{ where } \sigma_2 \to 0 \text{ as } x \to a.$$
$$\therefore\ f_1(x)f_2(x) = l_1l_2 + l_2\sigma_1 + l_1\sigma_2 + \sigma_1\sigma_2.$$
$$\therefore\ f_1(x)f_2(x) \to l_1l_2 \text{ as } x \to a,$$

since $\sigma_1 \to 0$, $\sigma_2 \to 0$, and $\sigma_1\sigma_2 \to 0$, under these circumstances, i.e.

$$\underset{x \to a}{\text{Lt}} \{f_1(x)f_2(x)\} = l_1l_2.$$

The theorem can be extended to any number of factors.

Theorem 3. Limit of a Reciprocal.

The limit of the reciprocal of a function of x is equal to the reciprocal of the limit of the function of x if this limit is not zero, i.e.

$$\underset{x \to a}{\text{Lt}} \left(\frac{1}{f(x)}\right) = \frac{1}{l}$$
$$\text{if } \underset{x \to a}{\text{Lt}}\ f(x) = l,$$

l not being zero.

As before, put

$$f(x) = l + \sigma, \text{ where } \sigma \to 0 \text{ as } x \to a.$$

$$\text{Then } \frac{1}{f(x)} = \frac{1}{l + \sigma},$$

and $\frac{1}{l + \sigma}$ approaches $\frac{1}{l}$ as $\sigma \to 0$.

$$\therefore\ \underset{x \to a}{\text{Lt}} \left(\frac{1}{f(x)}\right) = \frac{1}{l}$$
$$\text{if } \underset{x \to a}{\text{Lt}}\ f(x) = l.$$

Theorem 4. Limit of a Quotient.

The limit of the quotient of two functions of x is equal to the quotient of their limits, if these limits are finite, and if the limit of the denominator is not zero.

This theorem follows from Theorems 2 and 3 above, i.e.

$$\operatorname*{Lt}_{x \to a} \frac{f_1(x)}{f_2(x)} = \frac{l_1}{l_2}$$

$$\text{if } \operatorname*{Lt}_{x \to a} f_1(x) = l_1,$$

$$\text{and } \operatorname*{Lt}_{x \to a} f_2(x) = l_2, \ l_2 \text{ not being zero.}$$

WORKED EXAMPLES

1. Find $\operatorname*{Lt}_{x \to 0} \{a_0 + a_1x + \ldots + a_nx^n\}$

n being a finite positive integer, and the coefficients of x being all finite numbers.

$$\operatorname*{Lt}_{x \to 0} (a_0) = a_0$$
$$\operatorname*{Lt}_{x \to 0} (a_1x) = 0$$
$$\operatorname*{Lt}_{x \to 0} (a_2x^2) = 0$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad =$$

Adding, by Theorem 1,

$$\operatorname*{Lt}_{x \to 0} \{a_0 + a_1x + \ldots + a_nx^n\} = a_0.$$

2. Find the limit

$$\operatorname*{Lt}_{x \to 0} \frac{2x^2 + 3x^3 + x^4}{3x^2 + x^4 + x^6}.$$

Divide numerator and denominator by x^2.

$$\frac{2x^2 + 3x^3 + x^4}{3x^2 + x^4 + x^6} = \frac{2 + 3x + x^2}{3 + x^2 + x^4}.$$

$$\operatorname*{Lt}_{x \to 0} (2 + 3x + x^2) = 2 \text{ by Example 1 above.}$$

$$\operatorname*{Lt}_{x \to 0} (3 + x^2 + x^4) = 3 \quad \text{,,} \quad \text{,,}$$

$\therefore$ by Theorem 4, $\operatorname*{Lt}_{x \to 0}$ (given function) $= \frac{2}{3}$.

Circular Measure of Angles.

In theoretical investigations about angles and functions of angles, the angles are measured in *circular measure.*

Let $\angle$AOP be any angle (fig. 3). About O as centre, describe a circle APQ of any radius.

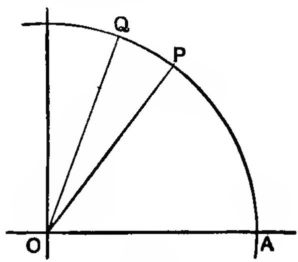


Fig. 3

By Euclid VI, xxxiii, the angle AOP is proportional to the arc AP on which it stands.

Hence any other angle AOQ must bear the same ratio to $\angle$AOP that the arc AQ bears to the arc AP; hence, if we make the arc AP = OA = r, and call the angle AOP the unit angle, then

$\angle$AOQ, measured in these units,

$$= \frac{AQ}{r} = \frac{s}{r}, \text{ if } s = AQ.$$

Putting θ for the angle AOQ, measured in these units,

$$\theta = \frac{s}{r}, \qquad \text{.........................(4)}$$

$$\text{i.e. } s = r\theta. \qquad \text{.........................(5)}$$

This unit angle is called the *radian*, so θ is the number which measures the $\angle$AOQ in *radians.*

To Convert from Radians to Degrees.

Consider the right angle AOQ (fig. 4).

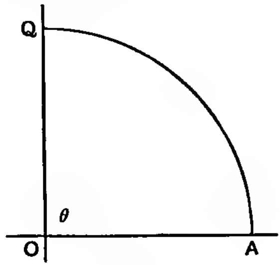


Fig. 4

$$\text{The arc AQ} = \tfrac{1}{4} \times 2\pi r = \frac{\pi r}{2}.$$

$$\therefore\ \theta = \frac{\pi r}{2}\Big/ r \text{ by (4)}$$

$$= \frac{\pi}{2},$$

i.e. the circular measure of a right angle is $\frac{\pi}{2}$ radians.

From this formula we can get any other angle by proportion thus:

Suppose we want the circular measure of 10°.

Let θ = the required measure,

$$\text{then as } 10^\circ : 90^\circ :: \theta : \frac{\pi}{2},$$

$$\text{i.e.}\quad 90 \times \theta = \frac{\pi}{2} \times 10,$$

$$\theta = \frac{10}{90} \times \frac{\pi}{2} = \frac{\pi}{18}.$$

In like manner equation (5) can be expressed in degrees.

The circular measure of $\psi°$ is

$$\frac{\psi}{180} \times \pi.$$

$$\therefore \frac{\psi}{180} \times \pi = \frac{s}{r},$$

$$\text{i.e.} \quad s = \left(\frac{\pi}{180}\right) r\psi. \quad \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

The formula is in its simplest form when the angle is expressed in circular measure, being then simply

$$s = r\theta.$$

The circular measure is the best measure in which to express angles in theoretical work, since mathematical formulæ involving angles take their simplest form when the angles are so expressed. In practical measurements the "degree" measure is used.

Tables for converting angles from one measure to the other are given in books of tables.

EXAMPLES

1. *Convert* 30°, 45°, *and* 90° *into circular measure.*
2. *Convert* 1, $\frac{1}{2}$, 1·54 *radians into degrees and minutes.*

Trigonometrical Limits.

The following relations are important:

$$(a) \quad \tan x > x > \sin x, \quad \ldots\ldots\ldots\ldots\ldots\ldots(7)$$

where x is the radian measure of a positive angle less than a right angle, i.e.

$$0 < x < \frac{\pi}{2}.$$

Let PQ be the arc of a circle of radius r (fig. 5), subtending an angle at O of $2x$. Let PT, QT be tangents at P and Q, meeting in T. Join PA and QA; then the quadrilateral OPAQ < sector OPAQ < quadrilateral OPTQ.

$$\therefore r^2 \sin x < r^2 x < r^2 \tan x.$$

$$\therefore \sin x < x < \tan x.$$

The reader will notice that this inequality is merely a mathematical statement of the *axiomatic* inequality:—

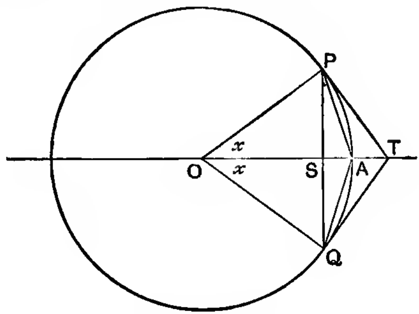


Fig. 5

The quadrilateral OPAQ < sector OPAQ < quadrilateral OPTQ. (See fig. 5.)

$$(b) \quad x > \sin x > x - \frac{x^3}{4}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

if x is positive.

Suppose x to be less than π. We have

$$\sin x = 2 \sin \tfrac{1}{2}x \cos \tfrac{1}{2}x,$$

$$\text{i.e. } \sin x = 2 \tan \tfrac{1}{2}x \cos^2 \tfrac{1}{2}x$$

$$= 2 \tan \tfrac{1}{2}x\,(1 - \sin^2 \tfrac{1}{2}x). \quad \ldots\ldots\ldots\ldots\ldots(9)$$

But $\tan \tfrac{1}{2}x > \tfrac{1}{2}x$,

and $\sin \tfrac{1}{2}x < \tfrac{1}{2}x$.

$\therefore \sin x > 2 \times \tfrac{1}{2}x\,\{1 - (\tfrac{1}{2}x)^2\}$,

i.e. $\sin x > x - \tfrac{1}{4}x^3$,

and $\sin x < x$ by inequality (a).

$\therefore x > \sin x > x - \tfrac{1}{4}x^3$.

We shall only use this inequality and the following one, when x is less than π.

$$(c) \quad 1 > \cos x > 1 - \tfrac{1}{2}x^2, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(10)$$

To prove this, we notice that

$$\cos x < 1.$$

This follows from the definition of cosine x.

$$\text{And } \cos x = 1 - 2\sin^2 \tfrac{1}{2}x$$
$$> 1 - 2\left(\tfrac{1}{2}x\right)^2 \text{ as } \sin \tfrac{1}{2}x < \tfrac{1}{2}x$$
$$> 1 - \frac{x^2}{2}.$$

To prove that $\underset{x \to 0}{Lt}\left(\dfrac{\sin x}{x}\right) = 1.$ ....(11)

By inequality (*b*) above

$$x > \sin x > x - \tfrac{1}{4}x^3.$$
$$\therefore\ 1 > \frac{\sin x}{x} > 1 - \tfrac{1}{4}x^2.$$
$$\therefore\ \underset{x \to 0}{\text{Lt}}\left(\frac{\sin x}{x}\right) = 1,$$

since the limit of 1 is 1, and the limit of $1 - \frac{1}{4}x^2$, as $x \to 0$ is 1 also.

To prove that $\underset{x \to 0}{Lt}\left(\dfrac{\tan x}{x}\right) = 1.$

$$\frac{\tan x}{x} = \frac{\sin x}{x} \cdot \frac{1}{\cos x}.$$
$$\therefore\ \underset{x \to 0}{\text{Lt}}\left(\frac{\tan x}{x}\right) = \underset{x \to 0}{\text{Lt}}\left(\frac{\sin x}{x}\right) \times \underset{x \to 0}{\text{Lt}}\left(\frac{1}{\cos x}\right).$$

Now $\underset{x \to 0}{\text{Lt}} \cos x = 1$ by (10) above.

$$\therefore\ \underset{x \to 0}{\text{Lt}}\left(\frac{1}{\cos x}\right) = 1.$$
$$\therefore\ \underset{x \to 0}{\text{Lt}}\left(\frac{\tan x}{x}\right) = 1 \times 1 = 1. \quad \text{..............(12)}$$

How the limit is approached.

$$\text{When } \underset{x \to a}{\text{Lt}} f(x) = l,$$
$$f(x) \to l,$$

whether $x \to a$ from values of $x < a$ or from values of $x > a$, if l is a normal limit.

Sometimes
$$\underset{x \to a}{\text{Lt}} f(x) = l_1$$

when x approaches a from values of $x < a$,

$$\text{and} \quad \underset{x \to a}{\text{Lt}} f(x) = l_2$$

when x approaches a from values of $x > a$.

The former is called a *progressive* limit, the latter a *regressive* limit.

$$\text{Normally,} \quad l_1 = l_2 = l.$$

The cases are illustrated in fig. 6.

In all the simple cases dealt with in this book the limit is approached normally. It has not been considered necessary to distinguish between

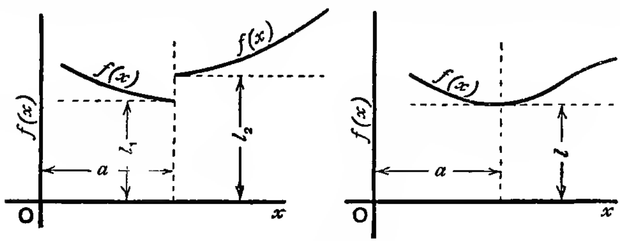


Fig. 6

the manners in which the limit is approached. Inspection of the proofs employed shows that they are equally valid whatever be the manner of approach.

Exercise 4

1. Find $\underset{x \to a}{\text{Lt}} x^2.$

2. Find $\underset{x \to a}{\text{Lt}} (\sin x).$

3. Find $\underset{x \to 1}{\text{Lt}} \left(\frac{x^2 + 3x - 4}{2x^2 + 6x - 8}\right).$

4. What are the circular measures of $43° 12'$, $52° 5'$, $107° 21'$?

5. What are the measures, in degrees and minutes, of the following angles, given in circular measure: $\frac{1}{3}$, 3, $5\frac{1}{2}$?

6. Show that $\underset{x \to 0}{\text{Lt}} \left(\frac{\text{arc tan } x}{x}\right) = 1.$

7. Show that $\underset{x \to 0}{\text{Lt}} \left[\frac{\sin \overline{y + x} - \sin y}{x}\right] = \cos y.$

[Combine $(\sin \overline{y + x} - \sin y)$ into a sine-cosine product of semi-sum and semi-difference of angles.]

8. Show that $\underset{x \to 0}{\text{Lt}} \left[\frac{\tan \overline{y + x} - \tan y}{x}\right] = \sec^2 y.$

9. Point out the flaw in the following argument:

$$\begin{aligned} \text{Let } x &= a. \\ \therefore x^2 &= ax. \\ \therefore x^2 - a^2 &= ax - a^2. \\ (x + a)(x - a) &= a(x - a). \\ \therefore x + a &= a. \\ \therefore a + a &= a. \\ \therefore 2 &= 1. \end{aligned}$$

CHAPTER V

Practical Applications

"It's no what we hae, but what we do wi' what we hae, that counts."—(OLD SCOTS PROVERB).

Beginners often have some difficulty in seeing why the calculus is so useful in the practical problems of physics and engineering. Nature is always changing, and things and their properties vary. Some of these properties are measurable — for instance, the relative positions of the heavenly bodies, the speed of trains, the temperature of bodies, the pressure of the atmosphere, the electric current in a wire, and so on. The quantities which measure these properties change, as the properties themselves change in the course of Nature, and, as we saw in Chapter I, the Calculus is the branch of mathematics which deals especially with changing quantities. The principle underlying its application may be seen in the following example.

In many cases first principles or experience show that some quantity (y) probably depends on another quantity (x). For instance, the distance a train moves depends on how long it has been moving. If s is the distance in feet travelled in an interval of time t sec., s is a function of t, i.e.

$$s = f(t). \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Everyone knows that the speed of the train comes into the problem. If the speed is fast, the train will move farther in a given time than if the speed is slow. If the speed is constant, the *definition* of speed leads at once to $s = vt$, where v is the speed in feet per second, and so $f(t) = vt$. If this were all that could happen the problem would be solved, but *the speed may vary from point to point in the path.* It is just this kind of difficulty that we have already encountered, and that the calculus enables us to overcome.

We have, on p. 59, arrived at the result

$$\frac{ds}{dt} = v,$$

where v is the velocity at the instant t. We can express *this fact by means of an integral*, see p. 75, i.e.

$$S = \int_{t_0}^{t_1} v dt, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

where S is the total distance travelled, in feet, in the interval $(t_1 - t_0)$ sec.

Can this integral be evaluated?

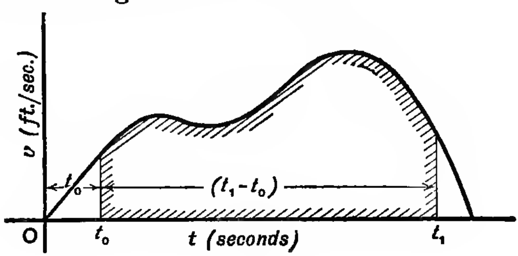


Fig. 1

If v is a known function of t, we may be able to integrate $\int_{t_0}^{t_1} v dt$ by the rules of the calculus, and get the result we wish. Or, if we have a graph of v in terms of t (fig. 1), then, since $\int_{t_0}^{t_1} v dt$ is the measure of the shaded area, the answer can be obtained graphically. This is a more powerful method than the first, but is longer.

If A is the area of the figure in square inches and the scales of the figure are

$$\begin{aligned} &1 \text{ in. vertically} &&= V \text{ ft. per second,} \\ &1 \text{ in. horizontally} &&= T \text{ sec.,} \\ \text{then } &1 \text{ sq. in.} &&= VT \text{ ft.} \end{aligned}$$

Hence $\int_{t_0}^{t_1} vdt = [VT] \times A = fA$, where f is the scale factor [VT].

EXAMPLES

1. If $v = at$, a being a constant

$$S = \int_{t_0}^{t_1} atdt$$

$= a \int_{t_0}^{t_1} tdt$, since a is a factor of every term in the sum denoted by $\int$, and may therefore be taken outside the $\int$ sign (see p. 80).

$= a \left[\frac{t^2}{2}\right]_{t_0}^{t_1}$ by Rule 2, p. 82,

i.e. $S = \frac{a}{2}[t_1^2 - t_0^2]$ ft.

2. Area of a Circle.

Let R be the radius of the circle in feet (fig. 2), and x the radius of a circle inside the bounding circle R.

Let A be the area of the circle up to radius x, in square feet.

Assume $A = f(x)$, where $f(x)$ is to be found, then ΔA is the increase in area arising from an increase Δx in x.

Let c be the circumference of the circle of radius x;

c_1, the circumference of the circle of radius $(x + \Delta x)$.

Then $c\Delta x < \Delta A < c_1\Delta x$.

$$\therefore \quad c < \frac{\Delta A}{\Delta x} < c_1.$$

$$\therefore \quad 2\pi x < \frac{\Delta A}{\Delta x} < 2\pi(x + \Delta x).$$

Finally, when $\Delta x \to 0$, c and c_1 do not differ sensibly, as the outer circle shrinks on to the inner one, i.e. $c_1 \to c$ as $\Delta x \to 0$:

$$\text{i.e. } \frac{dA}{dx} = c = 2\pi x.$$

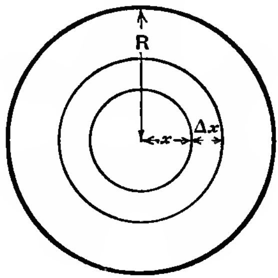


Fig. 2

(When we proceed to the limit ($\Delta x \to 0$), we drop into the d/dx notation.)

$\therefore$ $A = 2\pi\int_0^R x\,dx$; 2π is constant, and therefore is placed outside the sign of integration.

$$\therefore A = 2\pi\left[\frac{x^2}{2}\right]_0^R, \text{ by Rule 2, p. 82,}$$

$$= 2\pi\left[\frac{R^2}{2} - \frac{0^2}{2}\right]$$

$$= \frac{2\pi R^2}{2}$$

$$= \pi R^2 \text{ sq. ft.}$$

3. Motion under the Influence of Gravity.—*Required the distance, in feet, that a stone falls from rest in T sec. under the influence of gravity—neglecting air friction. The acceleration of gravity is 32·2 ft. per second per second (= g), a constant.*

Let v be the velocity of the stone at the time t sec., in feet per second, then $\frac{dv}{dt}$ is the acceleration of the stone at the time

t in feet per second per second, since dv/dt is the time-rate of change of velocity, which is what *acceleration* is, by definition.

$$\therefore \frac{dv}{dt} = g.$$

$$\therefore v = \int g dt.$$

The velocity at the end of any interval (t sec.) from the start is

$$v = \int_0^t g dt, \text{ since the stone starts from rest } (t = 0)$$

$$= \Big[gt\Big]_0^t \text{ by Rule 2, p. 82.}$$

$$= g[t - 0],$$

$$\text{i.e. } v = gt.$$

This formula gives us the velocity as a function of t.

Now $\frac{ds}{dt}$ is the velocity at t sec. from the start (see p. 59).

$$\therefore \frac{ds}{dt} = gt,$$

$$s = \int gt dt,$$

and the distance traversed from rest in T sec. is therefore

$$S = \int_0^T gt dt$$

$$= g\int_0^T t dt$$

$$= \frac{g}{2}\Big[T^2 - 0^2\Big] \text{ by Rule 2, p. 82.}$$

$$= \tfrac{1}{2} gT^2.$$

The formula $$S = \tfrac{1}{2} gT^2 = 16{\cdot}1T^2,$$

therefore gives the distance traversed, in feet, from rest in T sec.

4. Moments of Inertia.—Moments of inertia arise in problems of the bending of beams, rotational kinetic energy, and so on.

Routh defines the Moment of Inertia and Radius of Gyration as follows:—

"*If the mass of every particle of a material system be multiplied by the square of its distance from a straight line, the sum of the products so formed is called the 'moment of inertia' of the system about that line.*

"*If M be the mass of a system and k be such a quantity that Mk^2 is its moment of inertia about a given straight line, then k is called the 'radius of gyration' of the system about that line.*"

In the bending of beams the M. I. of plane areas is required.

Let the plane have mass, ρ lb. per square foot.

Let the moment of inertia be required about OY (fig. 3).

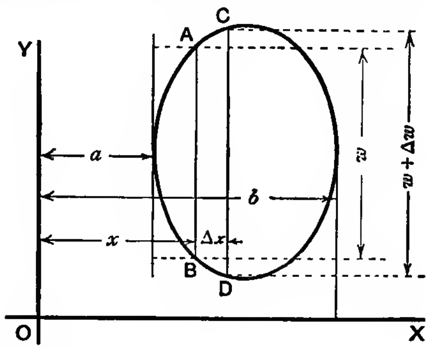


Fig. 3

Let w be the width (BA), in feet, of the plane at x, BA being parallel to OY.

Let I be the moment of inertia (lb.-ft.² units) of the area included between $x = a$ and $x = x$, and let $\text{I} = f(x)$, where $f(x)$ is to be found.

$$\begin{aligned}\Delta\text{I} &= \text{increase in I, due to an increase } \Delta x \text{ in } x,\\ &= \text{the increase in I, due to the area ABDC.}\end{aligned}$$

Now $w\Delta x <$ area ABDC $< (w + \Delta w)\Delta x$, if $w + \Delta w$ is the breadth at $(x + \Delta x)$,

$$\text{hence } \rho w \Delta x\, x^2 < \Delta\text{I} < \rho(w + \Delta w)\Delta x(x + \Delta x)^2,$$

$$\text{i.e. } \rho w x^2 < \frac{\Delta\text{I}}{\Delta x} < \rho(w + \Delta w)\,(x + \Delta x)^2.$$

Now when $\Delta x \longrightarrow 0$, $\Delta w \longrightarrow 0$,

and hence $\rho(w + \Delta w)(x + \Delta x)^2 \longrightarrow \rho w x^2$,

$$\text{i.e.} \quad \underset{\Delta x \longrightarrow 0}{\text{Lt}} \left(\frac{\Delta I}{\Delta x}\right) = \rho w x^2, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$$\text{i.e.} \quad \frac{dI}{dx} = \rho w x^2.$$

If I = whole moment of inertia,

$$I = \rho \int_a^b w x^2 dx, \text{ if } \rho \text{ is a constant.}$$

This result is in pound-foot units, if the density is expressed in pounds per square foot, and the dimensions of the surface are expressed in feet.

Note.—w cannot be put outside the sign of integration as it is not, in general, a constant.

No matter what shape of figure we assume, the conclusion is the same, namely, $I = \rho \int_a^b w x^2 dx$. We have only deduced the formula for fig. 3.

Special Case.

Find the moment of inertia of a uniform plane rectangle of density ρ lb. per square foot, about a line parallel to its shorter side and passing through the centre of the rectangle.

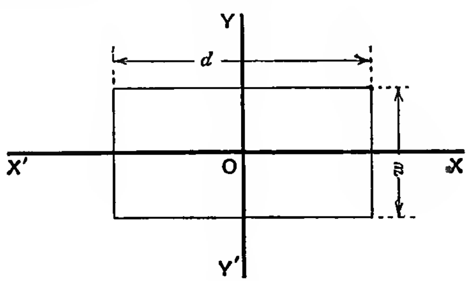


Fig. 4

Let d be the length in feet, and w the width in feet of the rectangle, d being greater than w (fig. 4).

Here $b = \frac{d}{2}$, $a = -\frac{d}{2}$, and w is constant, hence

$$I = \rho w \int_{-\frac{d}{2}}^{+\frac{d}{2}} x^2 dx = \rho w \left[\frac{x^3}{3}\right]_{-\frac{d}{2}}^{+\frac{d}{2}} \text{ by Rule 2, p. 82.}$$

$$= \rho w \left[\frac{d^3}{24} + \frac{d^3}{24}\right]$$

$$= \frac{\rho w d^3}{12}$$

$$= \rho w d \frac{d^2}{12}$$

$$= (\text{mass of rectangle}) \times \frac{d^2}{12},$$

i.e. $I = M\frac{d^2}{12}$ (lb.-ft.^{2}) where M is the mass of the rectangle in pounds.

This result is a case of Routh's rule, viz.:—

M.I. about an axis of symmetry

$$= \text{mass} \times \left[\frac{(\text{sum of squares of perp. semi-axes})}{3, 4, 5}\right]. \quad \ldots(4)$$

The denominator is to be 3 for a rectangle or square, 4 for an ellipse or circle, and 5 for an ellipsoid or sphere.

In the case above there is only one semi-axis perpendicular to the axis about which I is required, viz. the one perpendicular to Y′Y, i.e. $\frac{d}{2}$.

Routh's rule gives

$$I = M \times \frac{\left(\frac{d}{2}\right)^2}{3} = M\frac{d^2}{12},$$

which agrees with the result already found.

5. AN ELECTRICAL PROBLEM.—*Required the power lost in resistance—"I^2R" loss as it is called—in watts in a twin cable, the current in which is tapped off for lamps at i amperes per mile. The cable may be assumed to be very long compared to the distance between lamps, so that the current may be supposed to be tapped off continuously at i amperes per unit length.*

Measure x (fig. 5) from the far end.
The current passing down AB at P is ix.
The current passing down AB at p is $i(x + \Delta x)$.

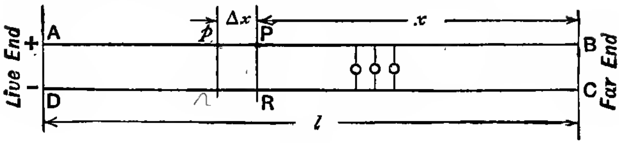


Fig. 5

Let r stand for the resistance in ohms, per mile, of *both* conductors AB and CD; then the resistance of the wires ($Pp + Rr$) is $r\Delta x$ ohms.

Let w stand for the loss in watts between the far end and the section PR, then the "I^2R" loss (Δw) in watts in Pp and Rr is given by the inequality

$$x^2 i^2 r \Delta x < \Delta w < i^2(x + \Delta x)^2 r \Delta x,$$

$$\text{i.e.} \quad x^2 i^2 r < \frac{\Delta w}{\Delta x} < (x + \Delta x)^2 i^2 r.$$

Now, as $\Delta x \rightarrow 0$, $(x + \Delta x)^2 \rightarrow x^2$.

$$\therefore \frac{dw}{dx} = i^2 x^2 r.$$

$$\therefore w = i^2 r \int_0^x x^2 dx = i^2 r \left[\frac{x^3}{3}\right]_0^x = \frac{i^2 r x^3}{3} \text{ watts.}$$

The whole current fed into the main is li amperes, and the whole resistance is rl ohms.

$$\text{Putting } I = li$$
$$\text{and } R = rl,$$

$$\therefore \text{the whole loss } W = \frac{l^2 i^2 r l}{3} \text{ (putting } x = l \text{ in } w)$$
$$= \tfrac{1}{3} I^2 R \text{ watts.}$$

Exercise 5

1. Why does the reference to time occur twice in specifying an acceleration?

Compare an acceleration of 1 mile per hour per second with one of 1 mile per second per hour.

2. A suburban electric train accelerates during the first 20 sec. of its run at $1\frac{1}{4}$ miles per hour per second; during the next 40 sec. at $\frac{1}{2}$ mile per hour per second. It then decelerates at 0·065 mile per hour per second for the next 120 sec. The brakes are then applied, and it decelerates at $1\frac{1}{2}$ miles per hour per second until it comes to rest.

Find how long the run takes, in seconds; find the length of the run, and hence find the average speed of the train during the run.

(Draw a velocity-time diagram and integrate it graphically.)

3. In question 2, at what speed is the horse-power taken by the train greatest? If the train weighs 70 tons, what is the maximum horse-power used?

4. An equilateral triangular flat plate is immersed vertically in water so that its upper apex is just submerged. Show that the total force on each face, due to hydrostatic pressure, is $\frac{1}{4}wl^3$, where w = weight of water per cubic foot and l = length of a side of the plate in feet.

5. Show that the moment of inertia of a flat circular plate, about an axis through its centre of gravity and perpendicular to the plate, is $Mr^2/2$ lb.-ft.^{2}, where M = mass of plate in pounds and r is the radius in feet.

(Calculate it from first principles and check your result by Routh's Rule.)

6. Show that the moment of inertia of an equilateral triangular plate, about an axis through an apex, in the plane of the plate and parallel to the opposite side, is $\frac{3}{8}Ml^2$ lb.-ft.^{2}, where M = mass in pounds and l = length of a side in feet.

7. Calculate the moment of inertia about XX′ of a plate of shape shown in fig. 6, the density of which is 1 lb. per cubic inch, and the thickness 1 in.

AB = 6 in.
AD = 1 in.
KL = 6 in.
GH = 1 in.
AK = KB.

State clearly the unit in which your answer is expressed. (Use Routh's Rule.)

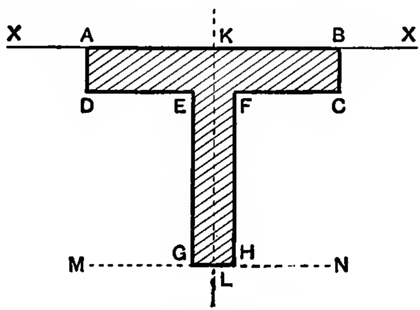


Fig. 6

8. Under the same conditions as in 7, what is the moment of inertia about the axis MN?

How do you account for the great difference in the moments of inertia?

9. Calculate the moment of inertia about XX′ for fig. 7, assuming unit thickness and unit density as in 7.

Calculate the moment of inertia about YY′.

10. What general conclusions do you draw from questions 7, 8, and 9 as to the distribution of material required to ensure a high moment of inertia? Does the usual design of a fly-wheel bear out your conclusions?

11. An ordinary fly-wheel weighs 1 ton and is 6 ft. in diameter. Assuming 85 per cent of the metal is in the

rim, which is 3 in. wide radially, what is its moment of inertia? Fix bounds within which your answer lies.

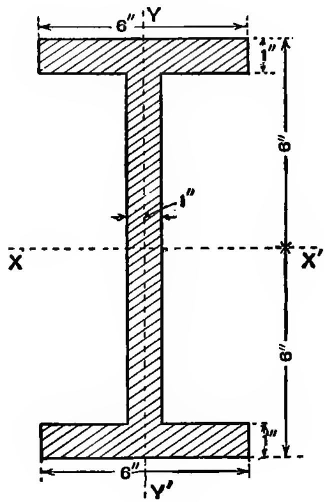


Fig. 7

12. The field magnet rotor of a large alternator is 5 ft. long, and 3 ft. in diameter, and may be treated as a solid cylinder weighing $7\frac{1}{2}$ tons. What is its moment of inertia, approximately?

13. Taking the earth as a sphere of diameter 8000 miles and of mass 6067×10^{18} tons, what is its moment of inertia about its polar axis? State carefully the unit in which you express your answer.

14. Current is tapped off a pair of street electric mains, of resistance r ohms per yard, at i amperes per yard. If the pressure across the mains at the sending end is E volts, what is the pressure at the far end, if l is the length of the pair of mains in yards ($2l =$ *total* length of copper conductor)?

CHAPTER VI

Differentiation and Integration

"Data aequatione fluentes quotcumque quantitates involvente fluxiones invenire et vice versa."—NEWTON.

In Chapter IV certain properties of limits and some theorems relating to them were deduced. These theorems furnish many important derivatives and integrals, and lead to several general theorems of differentiation and integration which are of the highest importance.

Differentiation of Sines and Cosines.

1. *To find the derivative of sin x.*

If $y = \sin x$,

$$\frac{dy}{dx} = \operatorname*{Lt}_{\Delta x \to 0} \left\{ \frac{\sin(x + \Delta x) - \sin x}{\Delta x} \right\}.$$

Now, $\sin(x + \Delta x) - \sin x$

$$= 2 \sin \tfrac{1}{2}\Delta x \cos(x + \tfrac{1}{2}\Delta x).$$

$$\therefore \frac{\sin(x + \Delta x) - \sin x}{\Delta x}$$

$$= \left[\frac{\sin \frac{1}{2}\Delta x}{\frac{1}{2}\Delta x}\right] \cos(x + \tfrac{1}{2}\Delta x).$$

$$\therefore \operatorname*{Lt}_{\Delta x \to 0}\left[\frac{\sin(x+\Delta x)-\sin x}{\Delta x}\right]$$

$$= \operatorname*{Lt}_{\Delta x \to 0}\left\{\left[\frac{\sin \frac{1}{2}\Delta x}{\frac{1}{2}\Delta x}\right]\cos(x+\tfrac{1}{2}\Delta x)\right\}$$

$$= \operatorname*{Lt}_{\Delta x \to 0}\left[\frac{\sin \frac{1}{2}\Delta x}{\frac{1}{2}\Delta x}\right] \times \operatorname*{Lt}_{\Delta x \to 0}\left[\cos(x+\tfrac{1}{2}\Delta x)\right]$$

by Theorem 2, p. 99.

$= 1 \times \cos x$, by equation (11), Chap. IV.

$$\therefore \frac{d}{dx}(\sin x) = \cos x, \quad \text{..........(1)}$$

$$\text{and} \int \cos x\, dx = \sin x. \quad \text{..........(2)}$$

2. *To find the derivative of cos x.*

If $y = \cos x$,

$$\frac{dy}{dx} = \operatorname*{Lt}_{\Delta x \to 0}\left[\frac{\cos(x+\Delta x)-\cos x}{\Delta x}\right]$$

$$= \operatorname*{Lt}_{\Delta x \to 0}\left[\frac{-2\sin(x+\frac{1}{2}\Delta x)\sin\frac{1}{2}\Delta x}{\Delta x}\right],$$

$$\text{whence } \frac{d}{dx}(\cos x) = -\sin x, \quad \text{........(3)}$$

$$\text{and} \int \sin x\, dx = -\cos x. \quad \text{........(4)}$$

(The reader should supply the details omitted above.)

Differentiation of Constants.

1. *Additive Constant.*

Let $y = f(x) + c$ where c is a constant number.

If x changes to $(x + \Delta x)$,

$$y + \Delta y = f(x + \Delta x) + c,$$
$$\text{and } y = f(x) + c.$$

Subtracting,

$$\Delta y = f(x + \Delta x) - f(x).$$
$$\therefore \frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

Proceeding to the limit when $\Delta x \rightarrow 0$, we find

$$\frac{dy}{dx} = \frac{d}{dx} f(x).$$

Rule III.—An additive constant may be ignored in finding derivatives.

2. *A constant multiplier.*

$$\text{Let } y = c \times f(x) \text{ where } c = \text{a constant number,}$$
$$y + \Delta y = c \times f(x + \Delta x).$$
$$\therefore \Delta y = cf(x + \Delta x) - cf(x)$$
$$= c[f(x + \Delta x) - f(x)].$$
$$\therefore \frac{\Delta y}{\Delta x} = c\left\{\frac{[f(x + \Delta x) - f(x)]}{\Delta x}\right\}.$$
$$\therefore \underset{\Delta x \rightarrow 0}{\text{Lt}} \left(\frac{\Delta y}{\Delta x}\right) = \underset{\Delta x \rightarrow 0}{\text{Lt}} c\left\{\frac{[f(x + \Delta x) - f(x)]}{\Delta x}\right\}$$
$$= c \times \underset{\Delta x \rightarrow 0}{\text{Lt}} \left[\frac{f(x + \Delta x) - f(x)}{\Delta x}\right],$$
$$\text{i.e. } \frac{dy}{dx} = c \times \frac{df(x)}{dx}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

Rule IV.—The derivative of c times a function of x is equal to c times the derivative of the function of x, or *a constant multiplier remains a constant multiplier.*

Differentiation of the Sum or Difference of Functions.

Let $y = \phi(x) + \psi(x)$, $\phi(x)$ and $\psi(x)$ being differentiable functions.

Then, as in proof of Rule 3, it easily follows that

$$\frac{dy}{dx} = \operatorname*{Lt}_{\Delta x \to 0}\left[\frac{\Delta\phi(x)}{\Delta x} + \frac{\Delta\psi(x)}{\Delta x}\right]$$

$$= \operatorname*{Lt}_{\Delta x \to 0}\left[\frac{\Delta\phi(x)}{\Delta x}\right] + \operatorname*{Lt}_{\Delta x \to 0}\frac{\Delta\psi(x)}{\Delta x}$$

by Theorem 1, p. 98.

$$= \frac{d}{dx}\phi(x) + \frac{d}{dx}\psi(x),$$

i.e. $$\frac{d}{dx}\{\phi(x) + \psi(x)\} = \frac{d}{dx}\phi(x) + \frac{d}{dx}\psi(x). \quad \ldots.(6)$$

Rule V.—The derivative of a sum (or difference) of functions is the sum (or difference) of the derivatives of each function, taken separately.

This follows from the fact that the theorem can obviously be extended to any *finite* number of functions.

Differentiation of Products of Functions.

Suppose $y = uv$ where $u = f(x)$, $v = \phi(x)$; and suppose both $f(x)$ and $\phi(x)$ are differentiable functions of x, so that the graph of each function is a smooth curve.

It is required to find $\frac{dy}{dx}$ for such an expression.

$$\Delta y = (u + \Delta u)(v + \Delta v) - uv$$

$$= v\Delta u + u\Delta v + \Delta u\Delta v.$$

$$\therefore \frac{\Delta y}{\Delta x} = u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x} + \Delta v\frac{\Delta u}{\Delta x}.$$

Proceeding to the limit where $\Delta x \rightarrow 0$,

$$\underset{\Delta x \rightarrow 0}{\text{Lt}} \frac{\Delta y}{\Delta x} = \underset{\Delta x \rightarrow 0}{\text{Lt}} \left(u\frac{\Delta v}{\Delta x}\right) + \underset{\Delta x \rightarrow 0}{\text{Lt}} \left(v\frac{\Delta u}{\Delta x}\right)$$
$$+ \underset{\Delta x \rightarrow 0}{\text{Lt}} \left(\Delta v\frac{\Delta u}{\Delta x}\right).$$

Since v is a continuous function, $\Delta v \rightarrow 0$ when $\Delta x \rightarrow 0$, whence by Theorems 1 and 2, p. 99,

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}. \quad\text{.................}(7)$$

Rule VI.—The derivative of the product of two functions is (1st function × derivative of 2nd function) plus (2nd function × derivative of 1st function).

The result can also be put

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{v}\frac{dv}{du} + \frac{1}{u}\frac{du}{dx},$$

on dividing (7) by $y = uv$.

This theorem can evidently be extended to the product of any finite number of functions.

EXAMPLE

$$\begin{aligned} y &= \sin x \cos x, \\ \sin x &= \text{1st function}, \\ \cos x &= \text{2nd function}. \\ \therefore \frac{dy}{dx} &= -\sin x \sin x + \cos x \cos x \\ &= \cos^2 x - \sin^2 x \\ &= \cos 2x. \end{aligned}$$

Differentiation of Quotients of Functions.

Let $y = \frac{u}{v}$, where y, u, and v are all differentiable functions of x, and v must not be zero.

Then $u = yv$.

By the last result, equation (7), we have

$$\frac{du}{dx} = v\frac{dv}{dx} + v\frac{dy}{dx}.$$

Multiply by $\frac{1}{v}$ and we get

$$\frac{1}{v}\frac{du}{dx} = \frac{y}{v}\frac{dv}{dx} + \frac{dy}{dx}.$$

$$\therefore \frac{dy}{dx} = \frac{1}{v}\frac{du}{dx} - \frac{y}{v}\frac{dv}{dx}$$

$$= \frac{1}{v}\frac{du}{dx} - \frac{u}{v^2}\frac{dv}{dx}$$

$$= \frac{1}{v^2}\left[v\frac{du}{dx} - u\frac{dv}{dx}\right]. \quad \text{.......(8)}$$

Rule VII.—To differentiate a quotient, multiply the denominator into the derivative of the numerator, and the numerator into the derivative of the denominator; take the latter product from the former and divide by the square of the denominator.

EXAMPLE

$$y = \frac{\sin x}{\cos x} = \tan x.$$

$$\text{Here } u = \sin x,\ v = \cos x.$$

$$\therefore \frac{dy}{dx} = \frac{\cos x \cos x + \sin x \sin x}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \sec^2 x.$$

Differentiation of a Function of a Function.

Suppose $y = f(u)$ and $u = \phi(x)$. Let $f(u)$ and $\phi(x)$ be differentiable functions, the first of u and the second of x.

The equation

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \times \frac{\Delta u}{\Delta x}$$

is formally true since we merely have to cancel out Δu.

Proceeding to the limit,

$$\underset{\Delta x \to 0}{\text{Lt}} \left(\frac{\Delta y}{\Delta x}\right) = \underset{\Delta x \to 0}{\text{Lt}} \left(\frac{\Delta y}{\Delta u}\right) \times \underset{\Delta x \to 0}{\text{Lt}} \left(\frac{\Delta u}{\Delta x}\right),$$

by Theorem 2, p. 99.

Now, when $\Delta x \to 0$, $\Delta u \to 0$ since $u = \phi(x)$ and $\phi(x)$ is continuous.

$$\therefore \frac{dy}{dx} = \underset{\Delta u \to 0}{\text{Lt}} \left(\frac{\Delta y}{\Delta u}\right) \times \frac{du}{dx}$$

$$= \frac{dy}{du} \times \frac{du}{dx}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

Rule VIII.—If y is a differentiable function of u, and u a differentiable function of x, the derivative of y with respect to x is the product of the derivative of y with respect to u into the derivative of u with respect to x.

EXAMPLE

$y = \cos(ax)$. Here $u = ax$. $\therefore y = \cos u$,

and $\dfrac{dy}{dx} = -\sin u \times a = -a \sin ax$.

Reciprocals.

The equation $\dfrac{\Delta y}{\Delta x} \times \dfrac{\Delta x}{\Delta y} = 1$ is formally true.

$$\text{Now } \underset{\Delta x \to 0}{\text{Lt}} \left[\frac{\Delta y}{\Delta x} \times \frac{\Delta x}{\Delta y}\right] = \underset{\Delta x \to 0}{\text{Lt}} \left(\frac{\Delta y}{\Delta x}\right) \times \underset{\Delta x \to 0}{\text{Lt}} \left(\frac{\Delta x}{\Delta y}\right),$$

provided both limits are finite.

Suppose y is a continuous function of x, then $\Delta y \to 0$ when $\Delta x \to 0$, hence

$$\underset{\Delta x \to 0}{\mathrm{Lt}}\left[\frac{\Delta y}{\Delta x} \times \frac{\Delta x}{\Delta y}\right] = \underset{\Delta x \to 0}{\mathrm{Lt}}\left(\frac{\Delta y}{\Delta x}\right) \times \underset{\Delta y \to 0}{\mathrm{Lt}}\left(\frac{\Delta x}{\Delta y}\right)$$

$$= \frac{dy}{dx} \times \frac{dx}{dy}.$$

Also $\underset{\Delta x \to 0}{\mathrm{Lt}}\left[\frac{\Delta y}{\Delta x} \times \frac{\Delta x}{\Delta y}\right] = 1$, since $\frac{\Delta y}{\Delta x} \times \frac{\Delta x}{\Delta y} = 1.$

$$\therefore \frac{dy}{dx} \times \frac{dx}{dy} = 1,$$

$$\text{i.e. } \frac{dy}{dx} = 1\Big/\frac{dx}{dy}, \quad \dots\dots\dots\dots\dots\dots (10)$$

provided $\frac{dx}{dy}$ is finite and not zero.

Constants in Integration.

(i) Suppose the integral $\int f(x)dx$ is required, and that $f(x)$ is given.

This integral is a function of x, $\phi(x)$, which has $f(x)$ as its derivative.

From this it follows that

$$\frac{d}{dx}\phi(x) = f(x). \quad \dots\dots\dots\dots\dots\dots (11)$$

Now, since the derivative of a constant quantity is 0, if (11) holds, then

$$\frac{d}{dx}[\phi(x) + c] = f(x). \quad \dots\dots\dots\dots\dots\dots (12)$$

Therefore all indefinite integrals must have a constant added to them.

This constant is entirely arbitrary. Its value depends on the conditions of each problem.

The constant need not be considered when a *definite* integral is taken.

$$\text{Let} \int f(x)dx = \phi(x) + c.$$

$$\text{Then} \int_{x_0}^{x_1} f(x)dx = \Big[\phi(x) + c\Big]_{x_0}^{x_1} = \phi(x_1) - \phi(x_0), \quad \ldots\ldots\ldots(13)$$

the constant going out in the difference.

(ii) Suppose $\int c \times f(x)dx$ is required, where c is a constant number.

$$\int cf(x)dx = \underset{\Delta x \to 0}{\text{Lt}} \Big\{\Sigma c\, f(x)\Delta x\Big\}, \text{ when } \Delta x \to 0$$

$$= c \times \underset{\Delta x \to 0}{\text{Lt}} \Big\{\Sigma f(x)\Delta x\Big\}.$$

(See numerical calculation, p. 81.)

$$\therefore \int cf(x)dx = c\int f(x)dx, \quad \ldots\ldots\ldots\ldots(14)$$

i.e. a constant multiplier (or divisor) can be placed outside the sign of integration.

Integral of the Sum of Two Functions.

$$\text{Since } \frac{d}{dx}\{\phi(x) + \psi(x)\} = \frac{d}{dx}\phi(x) + \frac{d}{dx}\psi(x), \text{ by (6)},$$

$$\therefore \phi(x) + \psi(x) = \int\left(\frac{d}{dx}\phi(x) + \frac{d}{dx}\psi(x)\right)dx.$$

$$\text{But} \quad \int\frac{d}{dx}\phi(x)dx = \phi(x),$$

$$\text{and} \quad \int\frac{d}{dx}\psi(x)dx = \psi(x).$$

$$\therefore \int\frac{d}{dx}\phi(x)dx + \int\frac{d}{dx}\psi(x)dx$$

$$= \int\left(\frac{d}{dx}\phi(x) + \frac{d}{dx}\psi(x)\right)dx. \quad \ldots\ldots(15)$$

Rule IX.—The integral of the sum (or difference) of (any finite number of) functions is the sum (or difference) of the separate integrals.

Integral of Product of Two Functions.

This theorem is highly important, and is known as "Integration of Products by Parts", or briefly "Integration by Parts".

By equation (7)

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx},$$ where u and v are differentiable functions of x.

$$\therefore \ uv = \int u\frac{dv}{dx}dx + \int v\frac{du}{dx}dx \text{ by (15).}$$

$$\therefore \int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx. \quad \ldots\ldots\ldots\ldots\ldots\ldots(16)$$

$$\text{Now } v = \int\frac{dv}{dx}dx \text{ and } u = \int\frac{du}{dx}dx.$$

Put w for $\frac{dv}{dx}$, a function of x,

then, by (16), $$\int uwdx = u\int wdx - \int\left(\frac{du}{dx}\times\int wdx\right)dx. \quad (17)$$

Rule X.—The integral of the product of two functions of *x* is equal to the first function × integral of second function, minus the integral of (the derivative of the first function multiplied by the integral of the second function).

EXAMPLE

Required $\int x \cos x \, dx = y$.

Here $u = x$, $v = \sin x$, since $\frac{dv}{dx} = \cos x$, if $v = \sin x$;

hence, regarding x as the first function and $\cos x$ as the second function, the rule gives directly

$$y = x \sin x - \int 1 \times \sin x dx$$
$$= x \sin x + \cos x.$$

(Check this result by differentiating it. Its derivative should, of course, be $x \cos x$.)

Derivative of x^n, when n is a fractional or negative index.

We have proved (p. 43) that

$$\frac{d}{dx}(x^n) = nx^{n-1}, \text{ when } n \text{ is a positive integer.} \quad (18)$$

Suppose n is a positive fraction $= \frac{p}{q}$.

$$\text{Put } y = x^{\frac{p}{q}}, \text{ i.e. } y \text{ is a function of } x.$$

$$\therefore y^q = x^p.$$

$$\therefore \frac{d}{dx}(y^q) = \frac{d}{dx}(x^p).$$

$$\therefore \frac{d}{dy}(y^q) \times \frac{dy}{dx} = \frac{d}{dx}(x^p).$$

$$\therefore qy^{q-1} \times \frac{dy}{dx} = px^{p-1},$$

since p and q are integers, and so (18) applies.

$$\therefore y^{q-1} \times \frac{dy}{dx} = \frac{p}{q}x^{p-1}.$$

$$\therefore (x^{\frac{p}{q}})^{q-1} \times \frac{dy}{dx} = \frac{p}{q}x^{p-1}.$$

$$\therefore \frac{dy}{dx} = \frac{p}{q} \times x^{\left\{p-1-\frac{p}{q}(q-1)\right\}}$$

$$= \frac{p}{q}x^{\frac{p}{q}-1}.$$

$$\therefore \frac{d}{dx}(x^n) = nx^{n-1}, \text{ when } x \text{ is a positive integer or fraction.}$$

Suppose n is a negative integer or fraction $= -m$,

$$\text{then} \quad x^{-m}x^m = 1.$$

$$\therefore \frac{d}{dx}(x^{-m}x^m) = 0.$$

$$\therefore x^m\frac{d}{dx}(x^{-m}) + x^{-m}\frac{d}{dx}(x^m) = 0.$$

$$\therefore x^m\frac{d}{dx}(x^{-m}) + mx^{-m}x^{m-1} = 0.$$

$$\therefore \frac{d}{dx}(x^{-m}) = -mx^{-m+m-1-m}$$

$$= -mx^{-m-1},$$

$$\text{i.e.} \quad \frac{d}{dx}(x^n) = nx^{n-1}, \qquad\qquad (19)$$

when n is a negative integer or a negative fraction.

We have now extended Rule I to make it apply to *any rational* index n, either positive or negative.

Exercise 6

Differentiate the following expressions:—

1. $3x^{\frac{7}{3}}$.
2. $x^3 + x^2 + x + 1$.
3. $(x + a)(x + b)$, a and b constants.
4. $x(1 + x^2)(1 + x^3)$.
5. x^{-n}.
6. $(x + a)/(x + b)$, a and b constant.
7. $\{x/(x + 1)\}^m$, m constant.
8. $\sqrt{1 + x^2}$.
9. $(2ax + x^2)^m$, a and m constant.
10. $x^3/\sqrt{1 + x^6}$.
11. $\tan x$. [Put $\tan x = \dfrac{\sin x}{\cos x}$. Also direct from limit, Exercise 8, p. 108.]
12. $\sec x$.

KEPLER (1571–1630)
From a contemporary painting

NAPIER (1550–1617)
From the portrait in the University of Edinburgh

DESCARTES (1596–1650)
From the portrait by Franz Hals in the Louvre

LEIBNITZ (1646–1716)
From the portrait in the Florence Gallery

MATHEMATICIANS AND PHILOSOPHERS

13. cosec x.
14. cot x.
15. sin $3x$.
16. sin $3x$ cos $2x$.
17. sin (cos x).
18. $(a + \sin x)/(b + \cos x)$, a and b constants.
19. $(a + \sin x)(b + \cos x)$, a and b constants.
20. $\sqrt{a + b \sin x}$, a and b constants.

Integrate the following functions of x:—

21. ax^3.
22. $\frac{a}{x^2}$.
23. $ax^{\frac{m}{n}}$.
24. $(ax^n + b)^m x^{n-1}$. [Put $ax^n + b = z$.]
25. $(ax + b)^m$.
26. 3 sin x.
27. sin x cos x.
28. sec x tan x.
29. 10 $\sec^2 x$.
30. sin $2x$.

CHAPTER VII

Infinitesimals

"Great fleas have little fleas upon their backs to bite 'em,
And little fleas have lesser fleas, and so *ad infinitum*.
And the great fleas themselves in turn have greater fleas to go on,
While these again have greater still, and greater still, and so on."—PROF. DE MORGAN.

Infinitesimals.

Up to the present $\frac{dy}{dx}$ has been defined as *one symbol* standing for

$$\underset{\Delta x \to 0}{\text{Lt}} \left(\frac{\Delta y}{\Delta x}\right),$$

and we have treated $\frac{d}{dx}$ as a single symbol just as $\sqrt{\ }$ is a single symbol.

In applied mathematics and in natural science generally, "d" (sometimes δ) is used in a different sense, so that dy, dx have definite meanings when standing separately.

This new usage arises from the fact that when we are dealing with quantities as distinct from numbers it is usually easy to see when one quantity is relatively small and small enough to be ignored.

In applied mathematics—indeed in all "concrete" mathematics as distinct from the mathematics of abstract numbers — mere numbers are not of much interest. It is not much use a farmer telling us that the stock on his farm is 100, unless he tells us whether he is speaking of bullocks or chickens.

Ascending Powers of Δx.

We have found that, if $y = f(x)$, Δy can often be expressed as an ascending power series in Δx.

For instance, if $y = x^4$,

$$\Delta y = 4x^3\Delta x + 6x^2(\Delta x)^2 + 4x(\Delta x)^3 + (\Delta x)^4. \ldots(1)$$

For any given value of x, the coefficient of powers of Δx reduce to common multipliers. Thus, if $x = 2$,

$$\Delta y = 32\,\Delta x + 24(\Delta x)^2 + 8(\Delta x)^3 + (\Delta x)^4. \ldots(2)$$

This equation is the *exact* expression for the increase (Δy) in y arising from an increase (Δx) in x, when the relation between x and y is $y = x^4$, and x has the special value 2 (see fig. 1).

Now, suppose we are considering changes in x, of the order 10^{-12}, i.e. $\Delta x = 10^{-12}$ say.

Δy is then the sum of

$$\begin{aligned} 32 \times 10^{-12} &= 32 \times 10^{36} \times 10^{-48} \\ 24 \times 10^{-24} &= 24 \times 10^{24} \times 10^{-48} \\ 8 \times 10^{-36} &= 8 \times 10^{12} \times 10^{-48} \\ 1 \times 10^{-48} &= 1 \times 10^{-48}. \end{aligned}$$

The *sum* of the last three terms is about 1 billionth of the first term.

If the first term ($32 \times 10^{36} \times 10^{-48}$) measures a "small quantity"* it is an *infinitesimal* of the first order.

* For discussion of this phrase, see p. 27.

The second term ($24 \times 10^{24} \times 10^{-48}$) is an infinitesimal of the second order. It is less than one billionth of the first order term.

The third term ($8 \times 10^{12} \times 10^{-48}$) is an infinitesimal of the third order. It is only one three-billionth

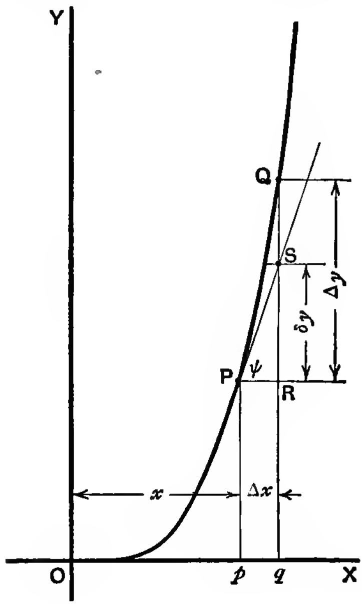


Fig. 1

part of the second order term, and is one four-billion-billionth part of the first term, and is therefore quite negligible compared to the first order term.

The fourth term (1×10^{-48}) is an infinitesimal of the fourth order, and is quite negligible compared to any of the others.

The result is that, in actually *calculating* Δy as a practical problem, *we need only retain infinitesimals of the lowest order*, provided Δx is small enough.

In the above case,

$$y = x^4$$

$$\text{and } \frac{dy}{dx} = 4x^3.$$

$$\therefore \Delta y = \left[\frac{dy}{dx} + f(\Delta x)\Delta x\right]\Delta x.$$

$$\therefore \Delta y = \frac{dy}{dx}\Delta x + [f(\Delta x)\Delta x]\Delta x. \quad \ldots(3)$$

It is evident from (1), then, that when Δx is very small, the important part of Δy is the $\left(\frac{dy}{dx}\Delta x\right)$ part.

The coefficient of the other term, $f(\Delta x)\Delta x$, approaches zero as $\Delta x \to 0$, and

$$\left[\frac{dy}{dx} + f(\Delta x)\Delta x\right] \to \frac{dy}{dx} \text{ as } \Delta x \to 0.$$

For this reason, $\frac{dy}{dx}\Delta x$ is given a special name, and is called the *differential* of y, and when Δx is small enough to make $f(\Delta x)\Delta x$ relatively negligible, Δx is called the *differential* of x. δy is written for the *differential* of y, and δx for the *differential* of x.

$$\text{Hence } \delta y = \frac{dy}{dx}\delta x, \quad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

where δx is any small increase of x.

We can proceed to the limit with δx if we wish, and get, dividing both sides of (4) by δx,

$$\frac{\delta y}{\delta x} = \frac{dy}{dx}$$

$$\text{and } \underset{\delta x \to 0}{\text{Lt}} \left(\frac{\delta y}{\delta x}\right) = \frac{dy}{dx} = \underset{\Delta x \to 0}{\text{Lt}} \left(\frac{\Delta y}{\Delta x}\right). \quad \ldots\ldots\ldots(5)$$

The term in equation (3), $[f(\Delta x)\Delta x]\Delta x$, plays no part in forming the limit of $\Delta y/\Delta x$, for

$$\underset{\Delta x \to 0}{\text{Lt}} [f(\Delta x)\Delta x] = 0, \quad \text{.............(6)}$$

and it contributes nothing of any importance to the change in y which accompanies *a sufficiently small* change in x.

It can therefore be omitted, and when it is omitted the differential or *infinitesimal* notation is used.

Fig. 1 shows clearly the principle involved in the use of infinitesimals.

$$\begin{aligned} \Delta y = RQ &= PR \tan \psi + SQ \\ &= \frac{dy}{dx}\Delta x + SQ \\ &= \delta y + SQ, \quad \text{...............(7)} \end{aligned}$$

and $SQ/\Delta x \to 0$ as $\Delta x \to 0$. Hence we may put

$$\text{change in } y \approx \frac{dy}{dx} \times (\text{change in } x), \quad \text{...(8)}$$

when Δx is small enough.

In applied mathematics and physics, it is usually not considered necessary to distinguish between δ and d in the notation. dx is often used instead of δx when an infinitesimal change in x is referred to. The beginner is advised to use one notation:

Δ for any change—large or small;
δ ,, an infinitesimal change;
and $\dfrac{d}{dx}$,, the limit notation.

d should not be used except in the limit notation.

The matter can be concisely put by using the formal definition of a limit given on p. 93.

If $\frac{\Delta y}{\Delta x}$ has a limit as $\Delta x \to 0$, this limit is written $\frac{dy}{dx}$.

Assuming the limit to exist, we must be able to find a finite positive number μ such that the inequality

$$0 < \left| \frac{\Delta y}{\Delta x} - \frac{dy}{dx} \right| < \kappa$$

holds so long as

$$0 < | \Delta x - 0 | < \mu$$

holds; where κ can be any finite positive number not zero. Now if

$$\frac{\Delta y}{\Delta x} = \frac{dy}{dx} + \epsilon, \quad \text{.....................(9)}$$

where ϵ is a positive or negative number, then

$$\left| \frac{\Delta y}{\Delta x} - \frac{dy}{dx} \right| = | \epsilon | .$$

$$\therefore \quad 0 < | \epsilon | < \kappa$$

$$\text{so long as } 0 < | \Delta x | < \mu.$$

These inequalities are the conditions that ϵ may have the limit zero as $\Delta x \to 0$, therefore if

$$\underset{\Delta x \to 0}{\text{Lt}} \left(\frac{\Delta y}{\Delta x}\right) \text{ exists and equals } \frac{dy}{dx},$$

we may put

$$\frac{\Delta y}{\Delta x} = \frac{dy}{dx} + \epsilon$$

where $\underset{\Delta x \to 0}{\text{Lt}} (\epsilon) = 0.$

Now, in any practical application of numbers to measure quantities we can always select a quantity q, such that any quantity less than q is negligible.

Hence, if we make $|\epsilon| < |q|$, we can write (9)

$$\frac{\Delta y}{\Delta x} \approx \frac{dy}{dx}, \qquad \text{(10)}$$

and, by sufficiently decreasing the interval in which $|\Delta x|$ must lie, we *can* always make $|\epsilon| < |q|$ for

$$|\epsilon| \to 0 \text{ as } |\Delta x| \to 0,$$

i.e. we write

$$\frac{\Delta y}{\Delta x} \approx \frac{dy}{dx}, \qquad \text{(11)}$$

when Δx is small enough.

Equation (11) must be read to mean that $\frac{\Delta y}{\Delta x}$ differs from $\frac{dy}{dx}$ by less than the arbitrary "tolerance" set when Δx is made sufficiently small, or, more briefly but more loosely,

$$\frac{\Delta y}{\Delta x} \text{ is nearly equal to } \frac{dy}{dx},$$

when Δx is small enough.

The "equals" Sign in Practical Calculation.

When physical or chemical *measurements* are in question, a slightly different interpretation is put on the sign " = " from that given to it in pure mathematics. In pure mathematics,

$$y = p, \qquad \text{(12)}$$

means that y is *exactly* equal to p. The number y is neither more nor less than the number p, in fact "=" means that y and p are identical numbers.

Now, suppose we say that l stands for the number which expresses the length of a straight line AB in terms of a unit of length, say the centimetre. If the line is measured and is found to measure say 10 cm., we say that

$$l = 10.$$

But it is almost certain that l does not equal 10 in the pure mathematical sense. Suppose the line to have been measured with a centimetre ruler. It is certain that the length is more than 9·9 cm. and less than 10·1 cm., so if we say that $l = 10$ in the pure mathematical sense we cannot be far wrong. If, later on, we found that a possible error of ± 0·1 cm. (which is ± 0·1 cm. in 10 cm., i.e. ± 1 per cent) is more than we ought to allow, then we should measure the line more accurately, say with a carefully-constructed travelling microscope fitted with vernier scales. We might then be sure that the length of our line is more than 9·99 cm. and less than 10·01 cm. The possible error in taking $l = 10$ is then limited to ± 0·01 cm. in 10, i.e. to 1 part in 1000; and so the process can be carried on, each approximation coming nearer to the true measurement as the possible error decreases.

The difficulty of making any physical measurement increases seriously as the permissible error decreases. To measure the line correct to ± 0·1 cm. may take one minute and require an appliance costing threepence. The same measurement correct to ± 0·01 cm. may take an hour and require apparatus costing many pounds; while, when a high degree of accuracy is required, such, for instance, as is attained in the measurement of base lines in geodesy (1 part in 2,000,000 say), the work takes years and becomes an international matter. Poincaré writes: ". . . this history (of geodesy) certainly teaches us what precautions must surround any serious scientific operation, and what time and trouble are involved in the conquest of a single new decimal".

It is unwise therefore to attempt to make a measurement to a higher degree of accuracy than is actually required. When we write $l = 10$ cm., we mean that 10 cm. can be taken as equal to l, the length of AB for the purposes of our problem, and not that 10 cm. is the *actual* length of the line.

It is in this sense that we may write, instead of (11),

$$\Delta y = \frac{dy}{dx}\delta x = \delta y, \quad \text{.............. (13)}$$

i.e. when Δx is small enough, it is unnecessary to distinguish between Δy and δy.

Approximate Calculations by Small Differences.

Suppose we wish to calculate the weight of a brass tube, per foot, correct to 5 per cent, the internal diameter being 2 in. and the thickness $\frac{1}{64}$ in.

d_1, internal diameter, $= 2$ in.
d_2, external diameter, $= 2\frac{1}{32}$ in.

$$\begin{aligned} W &= 12\rho \times (\text{area of cross-section of metal}) \\ &= 12\rho A, \end{aligned}$$

where W is the weight per foot in pounds,
ρ, the density in pounds per cubic inch,
and A, the area in square inches.

Let

A be the area of cross-section of tube in square inches,

c_1, the circumference of inner circle in inches,

c_2, ,, ,, outer ,, ,, (fig. 2).

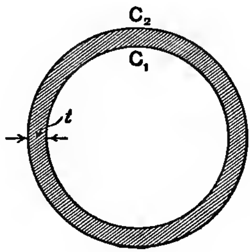


Fig. 2

Then, if t is the thickness in inches,

$$c_1 t < A < c_2 t.$$
$$\therefore \pi d_1 t < A < \pi d_2 t,$$
$$\pi d_1 t < A < \pi d_1 t\left(1 + \frac{2t}{d_1}\right),$$

since $d_2 = d_1 + 2t$.

Now,

$$\frac{2t}{d_1} = \frac{2 \times \frac{1}{64}}{2} = \frac{1}{64}.$$
$$\therefore \pi d_1 t < A < \pi d_1 t\left(1 + \frac{1}{64}\right).$$

π is not known accurately, but it is given by

$$3{\cdot}141 < \pi < 3{\cdot}142.$$
$$\therefore 3{\cdot}141 < \pi < 3{\cdot}141\left(1 + \frac{1}{3141}\right).$$

ρ is not known accurately; it varies with the composition of the brass, between 0·30 and 0·31.

$$\therefore\ 0{\cdot}30 < \rho < 0{\cdot}31,$$
$$\text{i.e. } 0{\cdot}30 < \rho < 0{\cdot}30\left(1 + \frac{1}{30}\right).$$

Combining all these inequalities,

$$12 \times 0{\cdot}30 \times 3{\cdot}141 \times 2 \times \frac{1}{64} < W$$
$$< \left(12\times 0{\cdot}30\times 3{\cdot}141\times 2\times\frac{1}{64}\right)\left(1+\frac{1}{30}\right)\left(1+\frac{1}{3141}\right)\left(1+\frac{1}{64}\right).$$

Now $\left(1 + \frac{1}{30}\right)\left(1 + \frac{1}{3141}\right)\left(1 + \frac{1}{64}\right) = \left(1 + \frac{1}{20{\cdot}07}\right)$.

If, therefore $\left(12 \times 0{\cdot}30 \times 3{\cdot}141 \times 2 \times \frac{1}{64}\right)$ is taken as the value of W, we can be sure that the result is within the permissible error of 5 per cent.

For tubes which are *sufficiently thin* compared to the diameter, the weight per foot may be calculated by $W = 12\rho\pi d_1 t$,

where ρ is the weight per cubic inch in pounds,
d_1, the internal diameter in inches,
and t, the thickness in inches.

The exact formula is

$$W = 12\rho\frac{\pi}{4}(d_2^2 - d_1^2)$$

$= 3\rho\pi(d_2^2 - d_1^2)$, where d_2 is the external diameter, and d_1 the internal diameter

$$= 3\rho\pi(d_2 + d_1)(d_2 - d_1)$$
$$= 12\rho\pi(d_1 + t)t.$$

There are undoubtedly many problems which can be calculated exactly without much difficulty; but there are also very many which cannot. Even if absolute exactness is possibly attainable, it is often only attainable at the cost of immense labour. To quote again from Poincaré's writings: "There we meet the physicist or the engineer who says, 'Will you integrate this differential equation* for me; I shall need it within a week for a piece of construction work that has to be completed by a certain date?' 'The equation,' we answer, 'is not included in one of the types that can be integrated, of which you know there are not very many.' 'Yes, I know; but then, what good are you?' More often than not a mutual understanding is sufficient. The engineer does not really require the integral (complete solution); he merely wants a certain figure which would be easily deduced from this integral if we knew it. Ordinarily we do not know it, but we could calculate the figure without it, if we knew just what figure and what degree of exactness the engineer required."

The above example (p. 142) has shown how, when the degree of exactness required is specified, the necessary calculation can be simplified. In the example, we reached a formula which gives sufficiently accurate results with a slide rule, whereas the exact formula cannot be calculated *directly* on a slide rule.

EXAMPLES FROM MECHANICS

1. *Find the position of the centre of gravity of a flat parabolic plate of uniform thickness, shaped as shown in fig. 3.*

Let AOB represent the plate. Choose the axes as shown in the figure, OX being the axis of symmetry. By symmetry the

* An example of a "differential equation" will be found on p. 184. We "integrate" a differential equation when we find the relation between the variables from which it arises.

C. G. must lie somewhere on OC. Let $\bar{x}$ stand for the distance of G, the C. G., from OY; then by taking moments, about OY, of the forces acting perpendicular to the plane of the plate, we have $\rho \times A \times \bar{x}$ = sum of moments of the forces on the pieces

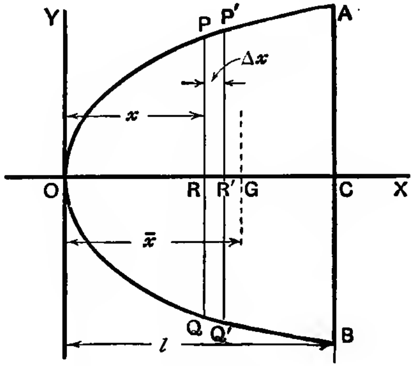


Fig. 3

of matter in AOB, about OY, where ρ is the weight per square inch in pounds and A the area in square inches. This follows from the *definition* of the centre of gravity as the point through which the resultant force of gravity acts.

Let x be the distance of any line PQ, at right angles to OC, from OY, and Δx the width of the strip PQQ′P′.

Then the area PQQ′P′ is greater than $QP \times \Delta x$, and less than $Q'P' \times \Delta x$.

Since OA is parabolic,

$$RP = y = \sqrt{4ax}, \text{ where } x = OR.$$

$$R'P' = y + \Delta y.$$

$$\therefore\ 2(y + \Delta y)\Delta x > PQQ'P' > 2y\Delta x.$$

$\therefore$ PQQ′P′ $= 2(y + \kappa\Delta y)\Delta x$, where κ is a positive fraction such that $0 < \kappa < 1$.

Hence the weight of the strip $= 2\rho(y + \kappa\Delta y)\Delta x$ lb.

Now the effective distance (d) of this strip from OY is given by

$$(x + \Delta x) > d > x.$$

$$\therefore\ d = x + \lambda\Delta x, \text{ where } 0 < \lambda < 1.$$

$\therefore$ moment of the weight of this strip

$$= 2\rho(y + \kappa\Delta y)\Delta x(x + \lambda\Delta x).$$

If M = moment of the portion of the plate OPQ,

$$\begin{aligned} \Delta M &= 2\rho(y + \kappa\Delta y)(x + \lambda\Delta x)\Delta x \\ &= 2\rho\{xy + \kappa x\Delta y + \lambda y\Delta x + \kappa\lambda\Delta y\Delta x\}\Delta x. \end{aligned}$$

Making the changes all infinitesimals, and dropping infinitesimals of second and higher orders, we get

$$\delta M = 2\rho xy\delta x.$$

$$\therefore\ \frac{\delta M}{\delta x} = 2\rho yx.$$

$$\therefore\ \underset{\delta x \to 0}{\text{Lt}} \left(\frac{\delta M}{\delta x}\right) = 2\rho yx.$$

$$\therefore\ \frac{dM}{dx} = 2\rho yx.$$

Hence, if M is the whole moment of the plate,

$$\begin{aligned} M &= 2\rho\int_0^l yxdx \\ &= 2\rho\int_0^l \sqrt{4ax}\,x\,dx \\ &= 2\rho\sqrt{4a}\int_0^l x^{\frac{3}{2}}dx \\ &= 2\rho\sqrt{4a}\left[\tfrac{2}{5}x^{\frac{5}{2}}\right]_0^l \\ &= 2\rho\sqrt{4a}\,\tfrac{2}{5}\,l^{\frac{5}{2}} \\ &= \tfrac{4}{5}\rho\sqrt{4al}\,l^2 \\ &= \tfrac{4}{5}\rho\,\overline{CA}\,l^2. \end{aligned}$$

Equating this quantity to $\rho A\bar{x}$, we get

$$\rho A\bar{x} = \tfrac{4}{5}\rho\,\overline{CA}\,l^2.$$

$$\therefore\ A\bar{x} = \tfrac{4}{5}\,\overline{CA}\,l^2.$$

$$\text{Now the area of the plate} = 2\int_0^l y\,dx$$
$$= 2\int_0^l \sqrt{4ax}\,dx$$
$$= 2\sqrt{4a}\int_0^l x^{\frac{1}{2}}dx$$
$$= 2\sqrt{4a}\left[\tfrac{2}{3}x^{\frac{3}{2}}\right]_0^l$$
$$= 2\sqrt{4a}\,\tfrac{2}{3}\,l^{\frac{3}{2}}$$
$$= \tfrac{4}{3}\sqrt{4al}\,l$$
$$= \tfrac{4}{3}\,\overline{\text{CA}}\,l.$$
$$\therefore\ \tfrac{4}{3}\,\overline{\text{CA}}\,l\,\bar{x} = \tfrac{4}{5}\,\overline{\text{CA}}\,l^2.$$
$$\therefore\ \bar{x} = \tfrac{3}{5}\,l,$$

i.e. the position of G is such that

$$\text{OG} = \tfrac{3}{5}\text{ of OC.}$$

2. *To find the moment of inertia of a uniform thin rectangular plate about an axis of symmetry.* (Compare p. 115.)

Fig. 4 illustrates this problem, Y′Y being the axis about which

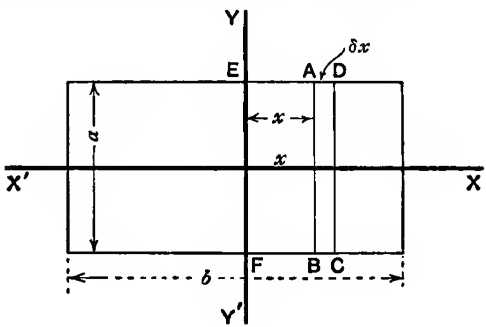


Fig. 4

the M. I. is required. Take the centre of the plate as origin and the axes of symmetry as the x and y axes.

Let x be the distance of AB, in feet, from the axis Y′Y. Consider the strip ABCD, which is δx ft. wide. The area of it is

$a\delta x$ sq. ft. and the mass $\rho a\delta x$ lb., if ρ is the density in pounds per square foot. This mass contributes to the M. I. something between $\rho a\delta x\, x^2$ and $\rho a\delta x\,(x + \delta x)^2$, say $\rho a\delta x\,(x + \kappa\,\delta x)^2$, where κ is a positive fraction between 0 and 1. This expression is

$$\rho a\delta x(x^2 + 2\kappa x\delta x + \kappa^2\delta x^2),$$
$$\text{i.e. } \rho a\delta x\, x^2 + 2\rho a\kappa x(\delta x)^2 + \rho a\kappa^2(\delta x)^3.$$

Retaining only the infinitesimal of the first order, we get, if I is the moment of inertia of the portion of the plate EFBA,

$$\delta I = \rho a x^2\delta x.$$
$$\therefore \frac{dI}{dx} = \rho a x^2.$$

The whole M. I. is given by

$$I = \rho a\int_{-\frac{b}{2}}^{+\frac{b}{2}} x^2 dx$$
$$= \rho a\left[\frac{x^3}{3}\right]_{-\frac{b}{2}}^{+\frac{b}{2}}$$
$$= \frac{\rho a b^3}{12}$$
$$= \left[\text{Mass} \times \frac{b^2}{12}\right] \text{lb.-ft.}^2.$$

Expansion of a Gas.

Let the pressure be related to the volume of a confined gas by a curve as shown in fig. 5.

When the volume is v c. ft. the pressure is p lb. per square *foot*.

When the volume is $(v + \Delta v)$ c. ft. the pressure is $(p + \Delta p)$ lb. per square *foot*.

Δp may be positive or negative; it is negative in the figure, if Δv is positive.

Suppose the gas expands by pushing a piston out very slowly against a pressure which is just less than the pressure of the gas.

Let A be the area of the piston in square feet.

The average pressure on the piston during the expansion Δv is something between p and $p + \Delta p$,

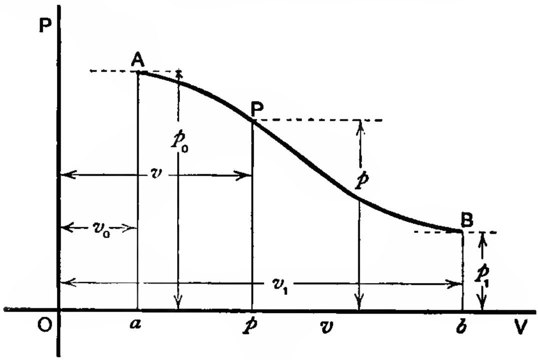


Fig. 5

say $p + \kappa\Delta p$, where κ is a positive fraction between 0 and 1.

The force on the piston is $(p + \kappa\Delta p)$A lb. The displacement is Δx ft.

The work done is $(p + \kappa\Delta p)A\Delta x$ ft.-lb.

But $A\Delta x = \Delta v$.

$\therefore$ $(p + \kappa\Delta p)\Delta v$ is the work done in the expansion Δv.

Suppose W is the work done in foot-pounds in expansion to volume v, then ΔW, the work done in foot-pounds in the expansion Δv, is given by

$$\begin{aligned} \Delta W &= (p + \kappa\Delta p)\Delta v \\ &= p\Delta v + \kappa\Delta p\Delta v. \end{aligned}$$

If the changes are infinitesimals, $\kappa\Delta p\Delta v$ is of the second order and $p\Delta v$ of the first.

Dropping the second order term,

$$\delta W = p\delta v, \text{ and proceeding to the limit,}$$

$$\text{i.e. } \frac{dW}{dv} = p,$$

$$\text{and } W = \int_{v_0}^{v_1} p\,dv \quad\text{.............................}(14)$$

$$= \text{work done in expanding from } v_0 \text{ to } v_1 \text{ in foot-pounds.}$$

∴ the work done in foot-pounds = (the area of A*ab*B) × *f*, where *f* is a scale factor.

This scale factor *f* can be found as follows:—

Suppose the p, v diagram—as a pressure graph, in terms of v, such as fig. 5 is called—is drawn to scales

1 in. vertically $\equiv$ P lb. per square inch.
1 in. horizontally $\equiv$ V c. ft.

Then 1 sq. in. of area on the diagram is equivalent to (P × 144 × V) ft.-lb.

Hence the "work" represented by A*ab*B is

$$(a \times P \times 144 \times V),$$

where a is the area of A*ab*B in square inches, i.e.

Work done in foot-pounds = (a in square inches) × *f* where $f = 144$ PV.

Indicator Diagrams.

Formula (14) is extremely important in engineering. An instrument called an *Indicator* is used to draw mechanically a p, v diagram for the steam on one side of a piston in an engine cylinder.

A is a small cylinder (fig. 6), B an accurately fitted piston of 1 sq. in. (say) in area, C a spring, D a joint by means of which the instrument can be attached to the engine cylinder, E a link mechanism by which the movement of the piston B is repeated at the pencil G, and F a drum on which a piece of glazed paper is carried. The drum F is given a rotational to and fro motion by means of a driving cord H, the free end of which is attached to the crosshead

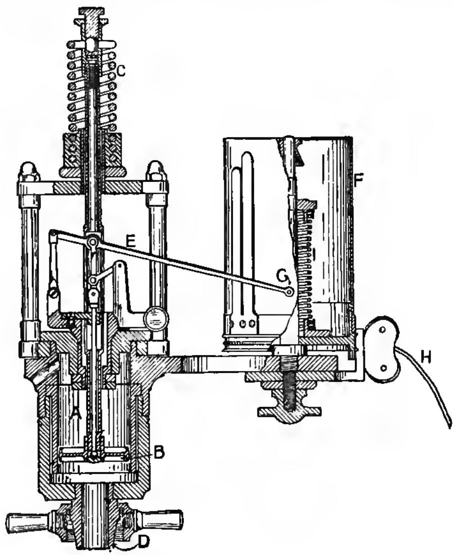


Fig. 6

of the engine. The string is kept taut by working against a spiral spring I contained in the drum F. The motion of F therefore repeats the outward and inward motion of the engine piston on a small scale; while the vertical motion of the pencil G, across the paper on F, repeats the motion of the piston B on a magnified scale. The pressure in the engine cylinder is the pressure on the steam side of B. The piston

B therefore responds, against the spring, to changes in the cylinder pressure, and the pencil G rises and falls as this pressure rises and falls.

The pencil therefore traces out a graph like that shown in fig. 7.

The curve is closed since the changes are cyclic and steady.

The area enclosed by this curve is by (14) a measure

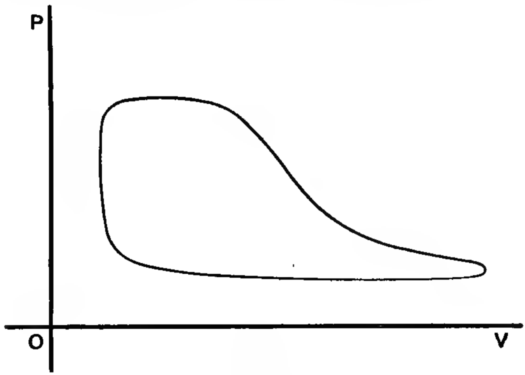


Fig. 7

of the actual work done by the steam in the cylinder on one side of the piston.

The appropriate scale factor f is known when the conditions under which the instrument is used are known, and hence the work done in foot-pounds $= f \times$ (area enclosed by the curve in square inches).

Of course any consistent units can be used by giving different values to f.

Exercise 7

1. Show that the moment of inertia of a thin rod of length l about an axis passing through one end and perpendicular to its length is

$$\frac{Ml^2}{3}, \text{ where M = mass of rod.}$$

Hence find the M. I. of a rod weighing 10 lb. and 7 ft. long.

2. Find the position of the centre of gravity of a semi-circular flat plate.

3. Find the position of the centre of gravity of a solid hemisphere of uniform density.

4. The period of oscillation of a simple pendulum is given by

$$T = 2\pi\sqrt{\frac{l}{g}}.$$

Show that the error in T introduced by a 1-per-cent error in g is equal and opposite to the error in T caused by a 1-per-cent error in l.

[Find the small change in T corresponding to a small change in l and in g. Make $\delta l = 0{\cdot}01\,l$ and $\delta g = 0{\cdot}01\,g$, and find the ratio of the corresponding small changes in T.]

CHAPTER VIII

Integration

"Nova stereometria doliorum; accessit stereometriæ Archimedeæ supplementum."—The Title of the first Book on the *Integral Calculus* by KEPLER (1615).

Some important methods for effecting the integration of functions have already been discussed. We propose to collect in this chapter the more important methods, and to give some examples of their applications.

1. By Differentiation.

When the derivative of a function is found, an integral formula follows at once. E.g.,

$$\frac{d}{dx}(\cos x \sin x) = \cos^2 x - \sin^2 x;$$

$$\therefore \int (\cos^2 x - \sin^2 x)dx = \cos x \sin x + c. \quad (1)$$

c is the arbitrary constant discussed on p. 128.

Sometimes it is easy to guess a function whose derivative is the given function to be integrated. If $\phi(x)$ is the given function and $f(x)$ the guessed function,

$$\int \phi(x)dx = f(x) + c, \quad \text{..............(2)}$$

$$\text{for } \phi(x) = \frac{d}{dx}f(x). \quad \text{...............(3)}$$

In this way certain *standard* elementary integrals are found. Many expedients have been devised for transforming integrals into simpler integrals which come within these standard forms. Skill in integration largely depends on ability to find readily suitable transformations which make the given integral depend on simpler known integrals. Each success adds another integral to the list of known forms, but in spite of the great number of integrals that are now known, functions still crop up which defy all attempts at integration. Sometimes, indeed, it can be foreseen that it is impossible to express a given integral exactly in terms of known functions, though we can always find the value of the integral approximately. Some of the simpler and more useful rules are given below.

2. Integration by Substitution.

Complicated expressions to be integrated can often be simplified by expressing them in terms of a new variable.

Suppose $\int_a^b \phi(x)dx$ is required.

We may suppose x to be a function of a new variable z, *provided that the new function of z chosen can assume all the values of x required in the integration.*

For instance, if $\int_1^3 x^n dx$ is required, we could not put $x = \sin z$ because $\sin z$ cannot be greater than 1.

But we could put $x = \tan z$, because a "z" can be found to give $\tan z$ equal to any number between 1 and 3.

It would not simplify the work, in this instance, to do this—the example is merely intended to show why certain functions are admissible while others are not.

Let $I = \int f(x)dx.$

Then $\frac{dI}{dx} = f(x) = \psi(z)$ if x is a function of z.

Let x be a differentiable function of z, say $\phi(z)$,

$$\text{then } \frac{dI}{dz} = \frac{dI}{dx}\frac{dx}{dz} \text{ by (9), p. 127}$$

$$= f(x)\frac{dx}{dz}.$$

$$\therefore I = \int\left\{\psi(z)\frac{d\phi(z)}{dz}\right\}dz. \quad \ldots\ldots\ldots(4)$$

This is the transformation formula for substitution.

EXAMPLES

(i) Evaluate $\int \sin(2x)dx$.

Put $2x = z$.

$$\therefore \frac{dx}{dz} = \tfrac{1}{2}.$$

$$\therefore I = \int \sin z \times \tfrac{1}{2} \times dz$$

$$= \tfrac{1}{2}\int \sin z dz$$

$$= -\tfrac{1}{2}\cos z = -\tfrac{1}{2}\cos(2x) + c.\text{*}$$

(ii) Evaluate $\int \sin(mx + n)dx$, m and n being constants.

Put $mx + n = z$.

$$\therefore x = \frac{z - n}{m}$$

$$\text{and } \frac{dx}{dz} = \frac{1}{m}.$$

* In all cases an arbitrary constant must be added to an indefinite integral. It will be understood in future.

$$\therefore\ I = \int\left(\sin z \times \frac{1}{m}\right)dz = \frac{1}{m}\int \sin z\,dz = -\frac{1}{m}\cos z$$

$$= -\frac{1}{m}\cos(mx+n).$$

(iii) Evaluate $\int \frac{1}{\sqrt{a^2 - x^2}}dx$, provided $-a < x < +a$.

Under these conditions it is admissible to put

$$x = a \sin z, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

for $\sin z$ may assume any value from -1 to $+1$.

Also $\frac{dx}{dz} = a\cos z$, by Rule 4, p. 123,

and $a^2 - x^2 = a^2 - a^2\sin^2 z.$

$$\therefore\ \sqrt{a^2 - x^2} = a \cos z.$$

$$\therefore\ \int \frac{dx}{\sqrt{a^2 - x^2}} = \int \psi(z)\frac{dx}{dz}dz = \int \frac{a\cos z}{a\cos z}dz$$

$$= \int dz = z.$$

But z = angle whose sine is $\frac{x}{a}$, by (5)

$$= \text{arc}\sin\frac{x}{a}.$$

$$\therefore\ \int \frac{dx}{\sqrt{a^2 - x^2}} = \text{arc}\sin\frac{x}{a}.$$

EXAMPLE.—Evaluate $\int \frac{dx}{a^2 + x^2}$, x being unlimited in value. (Put $x = a\tan z$.)

These are important integrals and should be remembered.

3. Integration by Parts.

This theorem has been discussed on p. 130. The rule is:

The integral of the product of two functions of x is the first function multiplied by the integral of the second, minus the integral of the derivative

of the first function multiplied by the integral of the second.

I.e.

$$\int\left\{u \times \left(\frac{dv}{dx}\right)\right\}dx = uv - \int\left\{\frac{du}{dx}v\right\}dx. \quad \ldots.(6)$$

EXAMPLES

(i) Evaluate $\int x \cos x \, dx.$

$$\text{Let } \int x \cos x \, dx = \text{I}.$$

The first function is $x = u$.

The second function is $\cos x = \frac{d}{dx}(\sin x)$, i.e. $v = \sin x$.

$$\text{Hence I} = \int x \frac{d}{dx}(\sin x)\, dx$$

$$= x \sin x - \int 1 \times \sin x \, dx, \text{ since } \frac{du}{dx} = 1 \text{ when } u = x$$

$$= x \sin x - \int \sin x \, dx$$

$$= x \sin x + \cos x.$$

(ii) Evaluate $\int \sin^2 x dx$.

$$\int \sin^2 x dx = \int (\sin x)\frac{d(-\cos x)}{dx}dx$$

$$= \sin x\,(-\cos x) - \int \cos x(-\cos x)dx$$

$$= -\sin x \cos x + \int (1 - \sin^2 x)dx$$

$$= -\sin x \cos x + x - \int \sin^2 x dx.$$

$$\therefore 2\int \sin^2 x dx = x - \sin x \cos x.$$

$$\therefore \int \sin^2 x dx = \frac{x - \sin x \cos x}{2}.$$

4. Integration by Reduction.

EXAMPLE

Integrate $\int \sin^n x dx$, n a positive integer.

$$\int \sin^n x dx = \int \frac{d}{dx}(-\cos x) \sin^{n-1} x dx = \mathrm{I}.$$

Integrating by parts, we have

$$\mathrm{I} = -\cos x \sin^{n-1} x + (n-1)\int \cos^2 x \sin^{n-2} x dx,$$

bearing in mind that

$$\frac{d}{dx}(\sin^{n-1} x) = (n-1)\sin^{n-2} x \cos x,$$

by Rule 8, p. 127.

$$\therefore \int \sin^n x dx = -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x dx$$

$$- (n-1)\int \sin^n x dx,$$

$$\text{i.e. } n\int \sin^n x dx = -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x dx,$$

$$\text{i.e. } \int \sin^n x dx = -\frac{\cos x \sin^{n-1} x}{n} + \frac{n-1}{n}\int \sin^{n-2} x dx \ldots\ldots(7)$$

This is a *formula of reduction* which makes the integral of $\sin^n x$ depend on that of $\sin^{n-2} x$, i.e. the degree of $\sin x$ is lowered by 2.

Repeated applications of the formula give any actual case required.

For instance, if $n = 4$,

$$\int \sin^4 x dx = -\frac{\cos x \sin^3 x}{4} + \frac{3}{4}\int \sin^2 x dx.$$

But $\int \sin^2 x dx$ can be obtained by the same reduction formula, putting $n = 2$ now.

$$\int \sin^2 x dx = -\frac{\cos x \sin x}{2} + \frac{1}{2}\int \sin^0 x dx$$

$$= -\frac{\cos x \sin x}{2} + \frac{1}{2}x.$$

$$\therefore \int \sin^4 x dx = \tfrac{1}{8}[3x - \sin x \cos x\,(2\sin^2 x + 3)]$$ on collecting up the terms.

If the integral is required between 0 and $\frac{1}{2}\pi$,

$$\int_0^{\frac{\pi}{2}} \sin^4 x dx = \frac{3}{8} \times \frac{\pi}{2} = \frac{3}{16}\pi$$

from the above result.

This result is a special case of

$$\int_0^{\frac{\pi}{2}} \sin^n x dx = \left[\frac{(n-1)(n-3)(n-5)\ldots 1}{n(n-2)(n-4)\ldots 2}\right] \times \frac{\pi}{2},$$

which is true when n is even. The reader should prove this direct from the reduction formula (7).

Inverse Trigonometrical Functions.

Such integrals as

$\int$ arc $\sin x dx$ are usually evaluated by substitution.

Put arc $\sin x = z$,

then $\sin z = x$.

$$\therefore \frac{dx}{dz} = \cos z.$$

$$\therefore \int \text{arc}\sin x dx = \int z \cos z dz$$

$$= z \sin z + \cos z,$$ by Ex. i, p. 159.

Hence $\int \text{arc}\sin x dx = x \text{ arc}\sin x + \sqrt{1 - x^2}.$

WORKED EXAMPLES

Freedom in integration can only be acquired by working out a large number of examples. The reader should study carefully the worked examples already given, and then work through the set of examples given at the end of the chapter. Practice in integration is given in Exercise 8, and the reader should do a few of these from time to time as he reads the rest of the book.

Hints for the solution of the more difficult of these examples are given in the Answers.

EXAMPLES

1. Integrate the following expressions by the method of substitution:—

(i) $\frac{1}{\sqrt{x}} \sin \sqrt{x}$.

(ii) $\frac{x}{\sqrt{1+x^2}}$.

(iii) $\sqrt{a^2 - x^2}$. $a > x$.

(iv) $\frac{1}{\sqrt{1-x^2}}$. $x < 1$.

(v) $\frac{1}{1+x^2}$.

(vi) $\frac{1}{x\sqrt{x^2-a^2}}$. $x > a$.

2. Integrate by parts the following expressions:—

(i) $x^2 \cos x$.

(ii) $x \cos 2x$.

(iii) $x^2 \arcsin x$.

(iv) $\frac{x \arcsin x}{(1-x^2)^{\frac{1}{2}}}$.

3. Show by reduction that

$$\int_0^{\frac{\pi}{2}} \sin^n x\,dx = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{4}{5} \cdot \frac{2}{3} \cdot \text{ when } n \text{ is odd.}$$

4. Show that

$$\int \sin^p\theta \cos^q\theta\,d\theta = -\frac{\sin^{p-1}\theta \cos^{q+1}\theta}{p+q} + \frac{p-1}{p+q}\int \sin^{p-2}\theta \cos^q\theta\,d\theta;$$

hence, by reduction, evaluate

$$\int_0^{\frac{\pi}{2}} \sin^4\theta \cos^2\theta\,d\theta.$$

Exercise 8a

Differentiate the following expressions:—

1. $a\sqrt{x}$.
2. $\sin(\cos x)$.
3. $\sin 3x \cos 2x$.
4. $\dfrac{1+x}{1+x^2}$.
5. $\sin x \cos x \tan x$.
6. $\text{arc}\tan x$.
7. $\dfrac{x}{\sqrt{a^2-x^2}}$.
8. $\sqrt{1-x^2} + \text{arc}\sin x$.
9. $\text{arc}\tan(n\tan x)$.
10. $\dfrac{\tan^3 x}{3} - \tan x + x$.
11. $\sqrt{\dfrac{1+x}{1-x}}$.
12. $\text{arc}\cos(4x^3-3x)$.
13. $\text{arc}\sin\dfrac{x+1}{\sqrt{2}}$.
14. $(a+x)^m(b+x)^n$.
15. $\text{arc}\sin(\sqrt{\sin x})$.
16. $\left(\dfrac{x}{1+x}\right)^n$.
17. $x\,\text{arc}\sin x$.
18. $\text{arc}\tan\sqrt{\dfrac{1-\cos x}{1+\cos x}}$.
19. $(a^2+x^2)\,\text{arc}\tan\dfrac{x}{a}$.
20. $\tan x\,\text{arc}\tan x$.
21. $\text{arc}\sec\left(\dfrac{1}{2x^2-1}\right)$.
22. $(x+a)\text{arc}\tan\sqrt{\dfrac{x}{a}} - \sqrt{ax}$.
23. $\text{arc}\tan\left(\dfrac{2x}{1-x^2}\right)$.
24. $\text{arc}\tan\left\{\dfrac{\sqrt{1+x^2}-1}{x}\right\}$.
25. $\text{arc}\cos\left(\dfrac{x^{2n}-1}{x^{2n}+1}\right)$.
26. $\text{arc}\sin\left(\dfrac{x}{\sqrt{1+x^2}}\right)$.
27. $\sin^3 x \cos x$.
28. $\dfrac{1}{\sqrt{2}}\,\text{arc}\sin\dfrac{x\sqrt{2}}{1+x^2}$.
29. $a(\sin x - \cos x)$.
30. If $y = a(z+\sin z)$ and $x = a(1-\cos z)$, show that

$$\frac{dy}{dx} = \frac{\sqrt{2ax-x^2}}{x}.$$

Exercise 8b

Integrate the following expressions:—

1. $ax^{\frac{m}{n}}$.
2. $(ax + b)^n$.
3. $\dfrac{1}{2 + 3x^2}$.
4. $x^2(a + x)^{\frac{1}{2}}$. [Put $a + x = t^2$.]
5. $x^2 (a + bx)$.
6. $\dfrac{1}{(a^2 + x^2)^{\frac{3}{2}}}$.
7. $(ax^3 + b)^2x^2$.
8. $\dfrac{1}{x^2(1+x^2)^{\frac{1}{2}}}$. [Put $x^{-2}+1=z^2$.]
9. $\sec^4 x$.
10. $x \sin x$.
11. x arc tan x.
12. $(a + bx + cx^2)(b + 2cx)$.
13. $x^3 (a + bx)$.
14. $\dfrac{x}{(1 - x)^3}$.
15. $(1 - \cos x)^2$.
16. $(ax^n + b)^m x^{n-1}$.
17. $$\int \tan^{2n}\theta d\theta = \frac{\tan^{2n-1}\theta}{2n-1} - \int \tan^{2n-2}\theta d\theta.$$

Prove this result: whence show that

$$\int \tan^{2n}\theta d\theta = \frac{\tan^{2n-1}\theta}{2n-1} - \frac{\tan^{2n-3}\theta}{2n-3} + \ldots - (-1)^n \tan\theta + (-1)^n\theta.$$

18. $x^4/(2 + 3x^2)$.
19. $(x + \sin x)/(1 + \cos x)$.
20. $x^4/(1 + x^2)^2$.
21. $\dfrac{x^2}{(x^2 + 1)(x^2 + 4)}$.
22. Show that

$$\int_0^{\pi} \sin mx \sin nx dx = 0, \text{ or } \frac{\pi}{2},$$

according as m and n are unequal or equal positive integers. [This result has important applications in Fourier's Theorem.]

23. Prove example 22 for cosine instead of sine.
24. $$\int_0^{\frac{\pi}{4}} \sec^4 \theta d\theta = \frac{4}{3}.$$

25. If $\phi(x) = \phi(a + x)$, show that

$$\int_0^{na} \phi(x)dx = n \int_0^a \phi(x)dx.$$

[Function has period a, whence result graphically.]

26. Show that $\phi(a)\,(b - a) < \int_a^b \phi(x)dx < \phi(b)\,(b - a)$,

if $\phi(x)$ increases steadily from a to b, and is positive.

What is the condition that $\phi(x)$ shall increase steadily from a to b?

27. Examine 26, when $\phi(x)$ *decreases* steadily from a to b.

28. Examine 26 for the case when $\phi(x)$ is negative between a and b, and when it changes sign between a and b.

29. If $\phi(x)$ is continuous, so that the graph of $\phi(x)$ is a continuous curve between $x = a$ and $x = b$, show that

$$\int_a^b \phi(x)dx = (b - a)\,\phi\,\{a + \theta\,(b - a)\},$$

where θ is a number which lies between 0 and 1.

[This result is known as the Mean Value Theorem.]

30. Find the approximate value of

$$\int_0^{\frac{\pi}{2}} \sqrt{1 - \tfrac{1}{9}\sin^2\theta}\; d\theta.$$

What is the maximum possible error in your answer?

This integral represents the length of a quadrant of an ellipse, whose major axis is 2, and whose eccentricity is $\frac{1}{3}$.

CHAPTER IX

Mensuration

"The quality of the human mind, considered in its collective aspect, which most strikes us, in surveying this record, is its colossal patience."—HOBSON ON "SQUARING THE CIRCLE".

Lengths of Curves.

Let the length of the curve AB be required (fig. 1).

Suppose $y = f(x)$ is the equation of AB. The length of AB is required between $x = x_0$ and $x = x_1$.

Divide ab into 9 equal parts. Erect ordinates

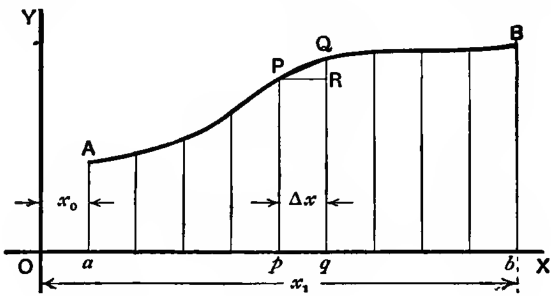


Fig. 1

at each point of division. Let pP, qQ be any two ordinates, distant Δx ft. apart. Then the chord PQ is nearly equal in length to the length of the curve between these ordinates; in fact, in the figure, the chord is scarcely distinguishable from the curve between P and Q. If *ab* were divided into 100 equal parts instead of 9, the chord PQ would be closer still to the curve between P and Q, and the more numerous the divisions the more difficult it becomes to distinguish the curve from the chord between any pair of ordinates.

The reader can soon convince himself of this by doing a little geometrical drawing, and finding how fine the division must be, for any given case, for the chord and the curve to become indistinguishable.

Fig. 2 shows the portion of the curve PQ on a larger scale.

Let PS and TQ be the tangent lines at P and Q, T being the point of intersection of these tangents.

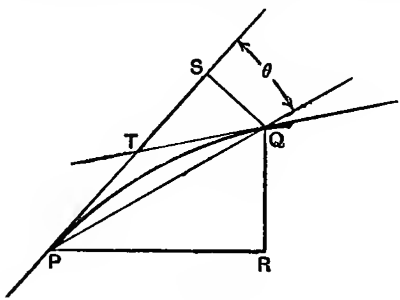


Fig. 2

Let QS be perpendicular to PS and let PQ make an angle θ with PS, then

$$\begin{aligned} \text{chord PQ} < \text{arc PQ} &< \text{PT} + \text{TQ} \\ &< \text{PT} + \text{TS} + \text{SQ} \\ &< \text{PS} + \text{SQ}. \end{aligned}$$

$$\therefore\ 1 < \frac{\text{arc PQ}}{\text{chord PQ}} < \cos\theta + \sin\theta.$$

Now as PR $\to$ 0, Q $\to$ P, and PQ approaches the tangent line PT, $\therefore \theta \to 0$.

$$\therefore \operatorname*{Lt}_{PR \to 0}\left(\frac{\text{arc PQ}}{\text{chord PQ}}\right) = 1.$$

If P is the point on the graph corresponding to x,

$$PR = \Delta x.$$

$$\therefore \operatorname*{Lt}_{\Delta x \to 0}\left(\frac{\text{arc PQ}}{\text{chord PQ}}\right) = 1.$$

Now, $$\frac{\text{arc PQ}}{\text{PR}} = \frac{\text{arc PQ}}{\text{chord PQ}} \times \frac{\text{chord PQ}}{\text{PR}},$$

and if s = length of curve from A to P,

$$\text{arc PQ} = \Delta s,$$
$$\text{and PR} = \Delta x.$$

$$\therefore \frac{\Delta s}{\Delta x} = \frac{\text{arc PQ}}{\text{chord PQ}} \times \frac{\text{chord PQ}}{\Delta x}.$$

$$\therefore \operatorname*{Lt}_{\Delta x \to 0}\left(\frac{\Delta s}{\Delta x}\right) = \operatorname*{Lt}_{\Delta x \to 0}\left(\frac{\text{arc PQ}}{\text{chord PQ}}\right) \times \operatorname*{Lt}_{\Delta x \to 0}\left(\frac{\text{chord PQ}}{\Delta x}\right).$$

$$\therefore \frac{ds}{dx} = 1 \times \operatorname*{Lt}_{\Delta x \to 0}\left(\frac{\text{chord PQ}}{\Delta x}\right).$$

$$\text{But chord PQ} = \sqrt{(\Delta y)^2 + (\Delta x)^2}.$$

$$\therefore \frac{\text{chord PQ}}{\Delta x} = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2},$$

$$\text{and } \operatorname*{Lt}_{\Delta x \to 0}\left(\frac{\text{chord PQ}}{\Delta x}\right) = \operatorname*{Lt}_{\Delta x \to 0}\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}$$

$$= \sqrt{1 + \operatorname*{Lt}_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right)^2}$$

$$= \sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

$$\therefore \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2},$$

$$\text{i.e. } s = \int_{x_0}^{x_1} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx, \quad \ldots\ldots(1)$$

when s is the length of curve between A and B, in feet—or any other unit of length which may be used.

Unfortunately, in many practical cases, this integral cannot be found by elementary methods, and even in simple cases devices are desirable to simplify the integral.

We will take one case which can be easily integrated from formula (1), and which is of some interest physically—the length of the cycloid between two cusps.

Note on Cycloid.—The cycloid is the curve generated by a fixed point on the circumference of a circle as the circle rolls without slipping along a straight line uv. It will therefore consist of a series of loops like uOv (fig. 3), with cusps at u, v, and so on. The length

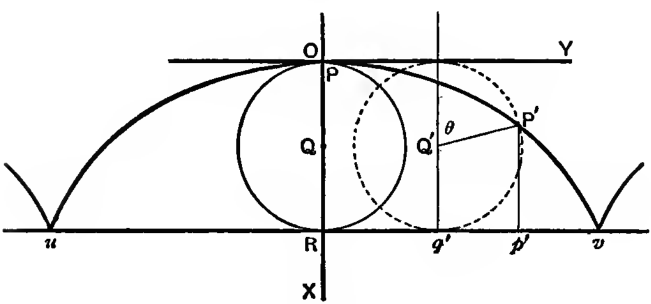


Fig. 3

of the curve uOv is, then, the length between two cusps, measured along the curve.

Let θ be the angle through which the radius QP has rotated in rolling from R to q'.

Then $Rq' = a\theta$, where a is the radius of the rolling circle—the generating circle as it is called,

$$q'p' = a \sin \theta,$$
$$p'P' = a(1 + \cos \theta).$$

Taking O as origin, OR and a line through O, perpendicular to OR, as axes of x and y, then if x and y are the co-ordinates of the point P′ on the cycloid,

$$y = Rp' = a(\theta + \sin \theta),$$
$$x = 2a - a(1 + \cos \theta) = a(1 - \cos \theta).$$

Eliminating θ, we get

$$y = a \text{ arc cos } \frac{a - x}{a} + \sqrt{(2ax - x^2)}. \quad \text{.......................(2)}$$

WORKED EXAMPLE

Find the length of a cycloid defined by the equation

$$y = a \text{ arc cos } \frac{a - x}{a} + \sqrt{(2ax - x^2)}$$

between $x = 0$ and $x = 2a$.

The curve is as sketched in fig. 4.

$$\frac{dy}{dx} = a\frac{d}{dx}\left(\text{arc cos } \frac{a - x}{a}\right) + \frac{d}{dx}\left(\sqrt{(2ax - x^2)}\right).$$

To differentiate the first term, we put

$$z = \text{arc cos } \frac{a - x}{a}.$$

$$\therefore \cos z = \frac{a - x}{a}.$$

$$\therefore a \cos z = a - x.$$

$$\therefore x = a(1 - \cos z).$$

$$\therefore \frac{dx}{dz} = a \sin z.$$

$$\therefore \frac{dz}{dx} = \frac{1}{a \sin z} = \frac{1}{\sqrt{a^2 - a^2 \cos^2 z}}$$

$$= \frac{1}{\sqrt{a^2 - (a - x)^2}}$$

$$= \frac{1}{\sqrt{2ax - x^2}}.$$

$$\therefore a\frac{d}{dx}\left(\text{arc cos } \frac{a - x}{a}\right) = \frac{a}{\sqrt{2ax - x^2}}.$$

For the second term, we get

$$\frac{d}{dx}(\sqrt{2ax - x^2}) = \frac{1}{2} \times \frac{1}{\sqrt{2ax - x^2}} \times (2a - 2x)$$

$$= \frac{a - x}{\sqrt{2ax - x^2}}.$$

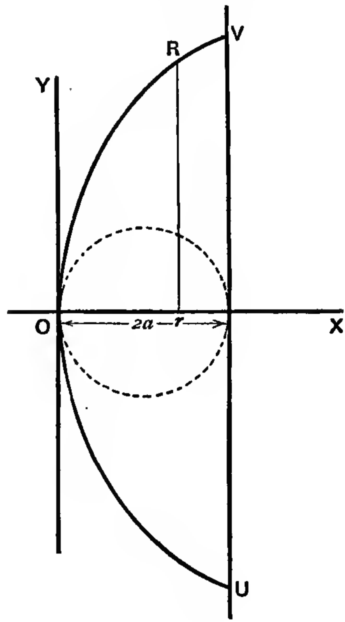


Fig. 4

Adding, we get

$$\frac{dy}{dx} = \frac{a}{\sqrt{2ax - x^2}} + \frac{a - x}{\sqrt{2ax - x^2}}$$

$$= \frac{2a - x}{\sqrt{2ax - x^2}} = \frac{\sqrt{2a - x}\,\sqrt{2a - x}}{\sqrt{2a - x}\,\sqrt{x}}$$

$$= \sqrt{\frac{2a - x}{x}}.$$

$$\therefore \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{\frac{2a}{x}}.$$

By (1), the integral giving the length we require is

$$\int_0^{2a}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx.$$

This integral is

$$\int_0^{2a}\sqrt{\frac{2a}{x}}\,dx = \left[\sqrt{2a}\times 2\sqrt{x}\right]_0^{2a} = 4a.$$

The whole length UOV $= 2 \times$ (length OV)

$= 8a.$ ....(3)

The length UOV is therefore four times the diameter of the generating circle. The interest of the cycloid lies in the fact that if a particle moves on a smooth cycloid under gravity, the vertex (O) of the cycloid being at the lowest point, the vibrations of the particle are truly *isochronous*, whence Lord Kelvin's name for harmonic motion, *cycloidal motion*. This example has been given to illustrate formula (1). Problems on the cycloid can be more easily worked by keeping the equations for x and y in terms of θ. We shall now discuss this method.

Use of a "Parameter", i.e. of a New Variable on which x and y depend.

Suppose x and y are each given as differentiable functions of another variable t.

$$s = \int\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx.$$

$$\therefore\ \frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2},$$

$$\text{and } \frac{ds}{dt} = \frac{ds}{dx}\,\frac{dx}{dt}$$

$$= \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,\frac{dx}{dt}.$$

$$\therefore\ s = \int\left\{\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,\frac{dx}{dt}\right\}dt.$$

$$\text{Now } \frac{\Delta y}{\Delta x} = \frac{\dfrac{\Delta y}{\Delta t}}{\dfrac{\Delta x}{\Delta t}},$$

and Δy and $\Delta x \to 0$ as $\Delta t \to 0$, since $x = f(t)$, and $y = \phi(t)$, and both $f(t)$ and $\phi(t)$ are continuous functions.

$$\therefore \underset{\Delta x \to 0}{\text{Lt}}\left(\frac{\Delta y}{\Delta x}\right) = \frac{\underset{\Delta x \to 0}{\text{Lt}}\left(\frac{\Delta y}{\Delta t}\right)}{\underset{\Delta x \to 0}{\text{Lt}}\left(\frac{\Delta x}{\Delta t}\right)}$$

$$= \frac{\underset{\Delta t \to 0}{\text{Lt}}\left(\frac{\Delta y}{\Delta t}\right)}{\underset{\Delta t \to 0}{\text{Lt}}\left(\frac{\Delta x}{\Delta t}\right)}.$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt}.$$

$$\therefore s = \int\left\{\sqrt{1 + \left(\frac{dy}{dt}\Big/\frac{dx}{dt}\right)^2}\frac{dx}{dt}\right\}dt$$

$$= \int\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}dt. \quad \ldots\ldots.(4)$$

EXAMPLE

To find the perimeter of a circle.

With the origin and co-ordinate axes, as shown in fig. 5, the co-ordinates of P are

$$y = a \sin \theta,$$
$$x = a \cos \theta,$$

where a is the radius of the circle.

Our t is here θ, and

$$\frac{dy}{dt} = \frac{dy}{d\theta} = a \cos \theta,$$

$$\text{and } \frac{dx}{dt} = \frac{dx}{d\theta} = -a \sin \theta,$$

$$\frac{dy}{dx} = \frac{a \cos \theta}{-a \sin \theta} = -\cot \theta,$$

$$\text{and } \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \cot^2\theta} = \sqrt{\frac{1}{\sin^2\theta}} = \frac{1}{\sin\theta},$$

$$\text{and } \frac{dx}{d\theta} = -a\sin\theta.$$

$$\therefore s = \int \frac{1}{\sin\theta} \times (-a\sin\theta)d\theta$$

$$= -a\int d\theta.$$

When $x = 0$, $\theta = \frac{\pi}{2}$; when $x = a$, $\theta = 0$.

$$\therefore \text{the length of a quadrant is } -a\Big[\theta\Big]_{\frac{\pi}{2}}^{0} = a\frac{\pi}{2}.$$

$\therefore$ the perimeter of the whole circle is $2\pi a$. ....(5)

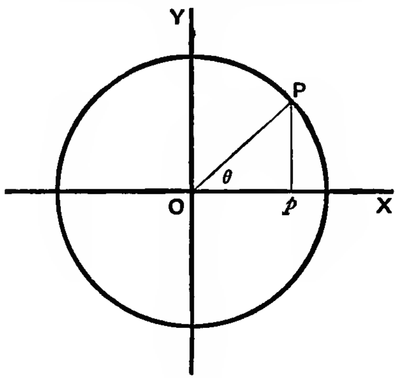


Fig. 5

The student should now rework by this method the problem of finding the length of the cycloid.

Areas.

Let AB (fig. 6) be a portion of a curve whose equation is

$$y = f(x),$$

and let the area under AB (A*ab*B) be required.

This area is

$$\int_{x_0}^{x_1} y\,dx. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

The problem of finding areas is therefore a direct problem of integration.

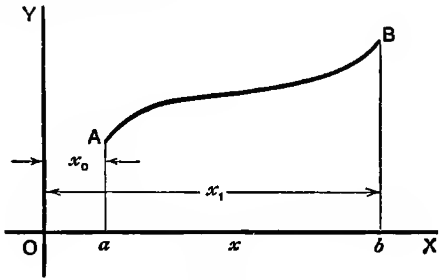


Fig. 6

WORKED EXAMPLE

Find the area of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1.$$

We have dealt with the geometry of the ellipse—so far as we need it—on pp. 8 to 12.

The ellipse has O as centre, and is shown in fig. 7.

It is symmetrical about OX and OY, so if the area of a quadrant is found, four times this area is the area required.

$$\text{Area of quadrant} = \int_0^a y\,dx.$$

By the given equation

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}.$$

$$\therefore y = b\sqrt{1-\frac{x^2}{a^2}}$$

$$= \frac{b}{a}\sqrt{a^2-x^2}.$$

$\therefore$ the area of a quadrant is $\frac{b}{a}\int_0^a \sqrt{a^2-x^2}\,dx.$

This integral has been given on p. 162, Ex. 1 (iii). It is

$$\left[\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\text{ arc sin }\frac{x}{a}\right].$$

$$\therefore \int_0^a \sqrt{a^2-x^2}dx = \left[\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\text{ arc sin }\frac{x}{a}\right]_0^a$$

$$= \frac{a^2}{2}\frac{\pi}{2}.$$

$$\therefore \text{ the area of the quadrant } = \frac{b}{a}\left[\frac{a^2}{2}\frac{\pi}{2}\right]$$

$$= \frac{\pi ab}{4},$$

and the area enclosed by the ellipse is πab. ....(7)

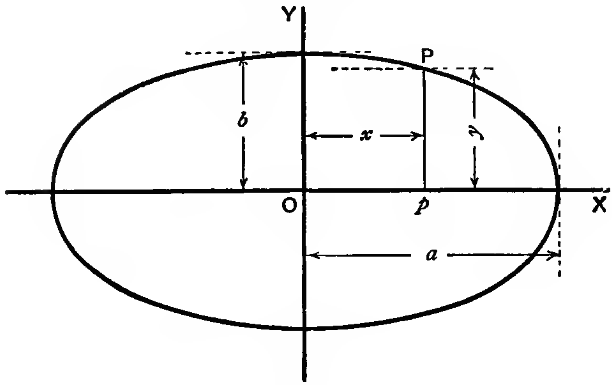


Fig. 7

Volumes of Solids of Revolution.

Suppose the volume of a figure (fig. 8) like ABCD is required, which is enclosed by two parallel planes and a curved boundary which is everywhere perpendicular to the end planes. The base can be divided into unit areas and includes, say, $10\frac{1}{2}$ of these. If the planes are 9 units of length apart, the volume is clearly $10\frac{1}{2} \times 9$ units of volume, i.e.

volume = area × height.

It follows that the volume of a cylinder is the area of the circular base multiplied by the length of the cylinder.

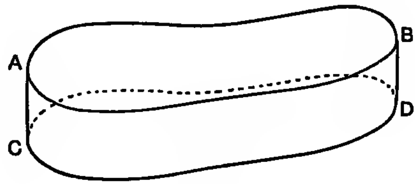


Fig. 8

A surface of revolution is the surface generated by a curve as it rotates about a fixed line in its plane.

Thus, if AB (fig. 9) rotates about *ab* so that every point in AB describes a circle whose plane is perpendicular to OX, and whose centre lies on OX, the surface swept out by AB is *a surface of revolution.*

Consider the volume enclosed by the surface formed by the revolution of AB (fig. 9) about OX and by the end planes $x = x_0$ and $x = x_1$.

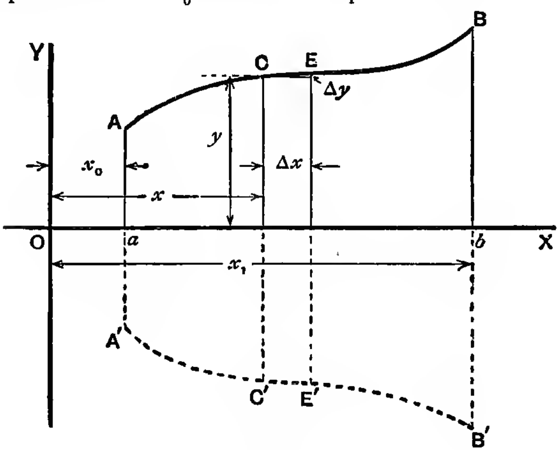


Fig. 9

Choose the origin and axes as shown. The dimensions shown on the figure then follow.

Let V be the volume AA′C′C between the planes

$$x = x_0 \text{ and } x = x.$$

Then the increase of volume, ΔV, in going from x to $x + \Delta x$ lies between $\pi y^2 \Delta x$ and $\pi(y + \Delta y)^2 \Delta x$, i.e., for the figure taken,

$$\pi y^2 \Delta x < \Delta V < \pi(y + \Delta y)^2 \Delta x.$$

$$\therefore \pi y^2 < \frac{\Delta V}{\Delta x} < \pi(y + \Delta y)^2.$$

$$\therefore \underset{\Delta x \to 0}{\text{Lt}} \left(\frac{\Delta V}{\Delta x}\right) = \pi y^2,$$

$$\text{i.e. } \frac{dV}{dx} = \pi y^2.$$

$$\therefore V = \pi \int_{x_0}^{x_1} y^2 dx. \quad \ldots\ldots.(8)$$

WORKED EXAMPLE

Find the volume of a sphere.

The volume of a sphere is enclosed by the surface of revolution of a circle about a diameter.

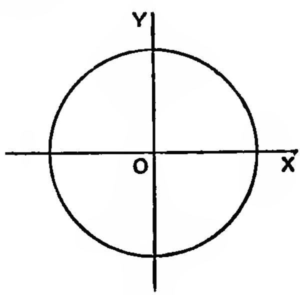


Fig. 10

Let O, the centre of the sphere, be taken as origin, and OX as the axis of revolution (fig. 10).

The equation of the revolving circle is

$$x^2 + y^2 = a^2 \text{ where } a \text{ is the rad. of sphere.}$$
$$\therefore y = \sqrt{a^2 - x^2}.$$

Let the circle spin about axis OX, then

$$\begin{aligned}
\text{vol. of sphere} &= 2 \times \text{vol. of } \tfrac{1}{2} \text{ sphere} \\
&= 2\int_0^a \pi y^2 dx \\
&= 2\pi\int_0^a (a^2 - x^2)dx \\
&= 2\pi\int_0^a a^2 dx - 2\pi\int_0^a x^2 dx \\
&= 2\pi\Big[a^2x - \tfrac{1}{3}x^3\Big]_0^a \\
&= \tfrac{4}{3}\pi a^3.
\end{aligned}$$

Vol. of a sphere, radius a, $= \frac{4}{3}\pi a^3$. ....(9)

Units.

The formulæ in this chapter have been given without specific units. This procedure has been followed because the question of units in mensuration is very simple.

Lengths can be in any unit [L] of length—say the inch. Areas are then measured in [L^2] units of area—say square inches, and Volumes are measured in [L^3] units of volume—say cubic inches.

Exercise q

1. Find the length of the semi-cubical parabola $y^2 = ax^3$ between $x = 0$ and $x = l$, y being positive.
2. Find the length of the curve $y^4 = ax^5$ between

$$x = 0 \text{ and } x = l,$$

y being positive.

3. Find the area above the x axis, included between the curves $y^2 = 2ax - x^2$ and $y^2 = ax$.

4. Plot the curve

$$y^2 = x^2\left(\frac{a^2 - x^2}{a^2 + x^2}\right) \text{ when } a = 10,$$

and find the area it encloses.

5. Find the area between the OX axis and the "witch"

$$y = \frac{2a}{x}\sqrt{2ax - x^2},$$

between $x = b$ and $x = a$ $(a > b > 0)$.

6. Show that the area contained by the curve

$$y = A_0 + A_1x + A_2x^2 + A_3x^3,$$

the OX axis, and the ordinates at x_0 and x_1, is given accurately by

$$\text{Area} = \frac{s}{3}(A + 2B + 4C),$$

where A, B, C, and s are obtained as follows:

Divide $(x_1 - x_0)$ into an even number (n) of equal parts, each equal to s.

A = sum of the end ordinates;
B = sum of the other odd ordinates (3rd, 5th, &c.);
C = sum of the even ordinates (2nd, 4th, &c.).

[Take $n = 2$, then prove the rule by repeated application of this case.]

This rule is known as *Simpson's Rule.*

When the rule is applied to find

$$\int_{x_0}^{x_1} y\,dx \text{ where } y = f(x),$$

the error, if any, is due to the deviation of $f(x)$ from an expression of type,

$$A_0 + A_1x + A_2x^2 + A_3x^3,$$

between the limits of integration. This deviation is usually quite small.

7. Find the area of the curved surface of a cone, of slant length l and radius of base r.

8. Find the area of the surface of a sphere of radius r.

9. Find the volume of a cone of height h and radius of base r.

10. Find the volume of the cap of a sphere of radius r and height h.

11. Find the volume of a paraboloid of revolution of height h and radius of base r.

12. Find the volume of a prolate spheroid of $2a$ major diameter and $2b$ minor diameter.

13. Find the volume of an oblate spheroid of $2a$ major diameter and $2b$ minor diameter.

14. Find the area of the surface of a sphere lying between two parallel planes distant a apart, the radius of the sphere being r.

CHAPTER X

Successive Differentiation

"... an ancient tale new told
And in the last repeating troublesome."—*King John.*

We saw in Chapter I that, if we start with a function of x, given by

$$y = f(x),$$

we can usually differentiate $f(x)$ and obtain a new function of x, $\psi(x)$, which is called the derivative or derived function. It is usual to denote this function by an accent, thus $f'(x)$ is written for $\psi(x)$, i.e.

$$\frac{d}{dx}f(x) = f'(x). \qquad\qquad (1)$$

This is, of course, a symbolic equation, and does not have any specific meaning until we specify $f(x)$.

If $\quad f(x) = x^4, f'(x) = 4x^3,$

and so on for other functions.

Now, there is no reason why we should not try to differentiate the new function just as we did the original one.

For instance, if

$$f(x) = x^4,$$
$$f'(x) = 4x^3, \text{ and}$$
$$\frac{d}{dx}(4x^3) = 12x^2.$$

Equation (1) then gives

$$\frac{d}{dx}\left\{\frac{d}{dx}f(x)\right\} = \frac{d}{dx}f'(x). \quad \ldots\ldots\ldots\ldots(2)$$

We can denote the new function

$$\frac{d}{dx}f'(x) \text{ by } f''(x)$$

in conformity with the notation $f'(x)$, so that

$$\frac{d}{dx}\left\{\frac{d}{dx}f(x)\right\} = f''(x). \ldots\ldots\ldots\ldots\ldots(3)$$

In the illustrative case above $f''(x)$ would be $12x^2$.

As a matter of *notation*, we can write $\frac{d^2}{dx^2}f(x)$ for

$$\frac{d}{dx}\left\{\frac{d}{dx}f(x)\right\},$$

hence

$$\frac{d^2}{dx^2}f(x) = f''(x). \ldots\ldots\ldots\ldots\ldots(4)$$

This equation is merely a matter of *notation*. We have not said that $\frac{d^2}{dx^2} = \frac{d}{dx} \times \frac{d}{dx}$. This equation would hold if d/dx were a symbol for a number, but it is not. It is a symbol for *an operation*, see p. 40.

$f''(x)$ is, *by convention*, the notation we use for the function obtained by differentiating $f(x)$ twice.

The function $f''(x)$ is called the *second derivative* of $f(x)$.

In the same way we can *express* the *third* derivative either as

$$\frac{d^3y}{dx^3} \text{ or as } f'''(x),$$

and the nth derivative either as

$$\frac{d^ny}{dx^n} \text{ or } f^{(n)}(x).$$

This process is what is known as *successive differentiation.*

Sometimes it is convenient to use the following notation:—

$$\left.\begin{array}{l} \text{If } y = f(x) \\ \quad \text{the 1st derivative is written } y_1, \\ \quad \text{,, 2nd \quad ,, \quad ,, \quad } y_2, \\ \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \quad \text{,, } n\text{th \quad ,, \quad ,, \quad } y_n. \end{array}\right\} \cdots\cdots\cdots\cdots(5)$$

WORKED EXAMPLES

1. *Find y_n, if $y = \sin(ax + b)$.*

$$y_1 = a\cos(ax+b) = a\sin\left(ax+b+\frac{\pi}{2}\right),$$

$$y_2 = a^2\sin\left(ax+b+\frac{2\pi}{2}\right),$$

$$y_3 = a^3\sin\left(ax+b+\frac{3\pi}{2}\right),$$

$$\cdot \quad \cdot = \cdot \quad \cdot \quad \cdot \quad \cdot,$$

$$\text{and } y_n = a^n\sin\left(ax+b+\frac{n\pi}{2}\right).$$

2. *If $y = ax \sin x$, prove that*

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + (x^2+2)y = 0.$$

We are given $y = ax\sin x$.

$$\therefore\ y_1 = a\sin x + ax\cos x,$$

$$\text{and } y_2 = 2a\cos x - ax\sin x.$$

$$\therefore\ x^2y_2 = 2ax^2\cos x - ax^3\sin x,$$

$$-2xy_1 = -2ax\sin x - 2ax^2\cos x,$$

$$x^2y = ax^3\sin x,$$

$$2y = 2ax\sin x.$$

Adding, $\quad x^2y_2 - 2xy_1 + x^2y + 2y = 0,$

$$\text{i.e. } x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + (x^2+2)y = 0.$$

This last equation is called a *differential equation.**

* For some further information on *differential equations,* see p. 248.

A solution of it is clearly $y = ax \sin x$, for this value of y satisfies the equation.

3. *Suppose $f(x)$ can be expressed as the sum of a series* $A_0 + A_1x + A_2x^2$, *where* A_0, A_1, *and* A_2 *are constants. It is required to express* A_0, A_1, *and* A_2 *in terms of derivatives of any orders which may be necessary.*

$$\text{Put } f(x) = A_0 + A_1x + A_2x^2, \quad \ldots\ldots\ldots\ldots\ldots (a)$$
$$\text{then } f(0) = A_0.$$

Differentiate equation (a), then

$$f'(x) = A_1 + 2A_2x. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (b)$$
$$\therefore f'(0) = A_1.$$

Differentiate (b), then

$$f''(x) = 2A_2.$$
$$\therefore f''(0) = 2A_2.$$
$$\therefore f(x) = f(0) + f'(0)x + \frac{f''(0)}{1.2}x^2. \quad \ldots\ldots\ldots (c)$$

We have thus expressed the function as a power series in x, the coefficients of the terms of which depend on the values of the successive derivatives of $f(x)$ at $x = 0$. Thus, if

$$f(x) = (a + x)^2,$$
$$\text{then } f'(x) = 2(a + x),$$
$$\text{and } f''(x) = 2.$$
$$\therefore f(0) = a^2,$$
$$f'(0) = 2a,$$
$$f''(0) = 2,$$

and $(a + x)^2 = a^2 + 2ax + x^2$ on substituting these values of $f(0)$, $f'(0)$, and $f''(0)$ in equation (c).

The appearance of the higher derivatives in

(*a*) Differential Equations,
and (*b*) Series,

should be carefully noted. (See Exer. 10.) They are constantly appearing in these subjects, and a clear idea of their nature and of why they appear is essential.

Exercise 10

1. Let $y = x^4 + x^3 + x^2 + x + 1$.

Show that $\dfrac{d^5y}{dx^5} = 0$.

2. If $y = x^2 \sin ax$, a being constant, find y_3.

3. If $y = \sin mx \cos nx$, m and n being constants, find y_3.

4. Prove that the nth derivative of x^n, when n is a positive integer, is $n!$ (i.e. *factorial* $n = n(n-1)(n-2) \ldots 2.1$).

5. If $y = A \sin mx + B \cos mx$, show that

$$\frac{d^2y}{dx^2} + m^2y = 0,$$

where m is a given constant and A and B arbitrary constants.

This is a most important result, and should be remembered. It is the *differential equation* of harmonic motion, and is perpetually occurring in physics and engineering.

6. Show that $\tan x$ satisfies the equation

$$\frac{d^2y}{dx^2} = 2y\frac{dy}{dx}.$$

7. If $y = \dfrac{1}{x}$,

$$\text{show that } \frac{d^ny}{dx^n} = (-1)^n\, 1.2.3 \ldots n x^{-(n+1)}$$
$$= (-1)^n\, n!\, x^{-(n+1)}.$$

8. If $y = A + Bx + Cx^2 + Dx^3 + \ldots Nx^n$, n finite,

show that
$$A = y_0,$$
$$B = \left(\frac{dy}{dx}\right)_0,$$
$$C = \frac{1}{1.2}\left(\frac{d^2y}{dx^2}\right)_0,$$
$$\ldots = \ldots\ldots$$
$$N = \frac{1}{n!}\left(\frac{d^ny}{dx^n}\right)_0,$$

where the suffix o denotes that the value of each derivative is to be taken at $x = 0$.

[Differentiate $y = A + Bx + Cx^2 \ldots$ successively, and put $x = 0$ in the equations

$$\frac{dy}{dx} = B + 2Cx + \ldots,$$

&c.]

9. We can therefore write (Example 8),

$$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \frac{f^{(n)}(0)x^n}{n!}$$

[$f^{(n)}(0)$ stands for the value of $\frac{d^ny}{dx^n}$, when x is put equal to zero], provided $f(x)$ can be expanded in a finite ascending series of powers of x.

The necessary condition for this to hold is that $\frac{d^{n+1}y}{dx^{n+1}}$, and hence all higher derivatives, must be zero.

Examine the case when $f(x) = x^5$.

10. Put $x = a + y$ in the expansion in (9) where a is a constant, and find the expansion for $f(a + y)$ in powers of y, by the method of p. 185.

11. Show, by using the expansion found in (10), that

$$(a + x)^4 = a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + x^4.$$

12. If $y = (1 + x + x^2)^4$, expand y in ascending powers of x.

CHAPTER XI

Curvature of Plane Curves

"Curvature is the 'moreness' of slope per inch of 'go forward'."

We shall suppose that all the curves discussed in this chapter lie in one plane—the plane of the paper.

A curve has, as a rule, a definite direction at every point in it, and, at any particular point, may be said to "point" in a direction which is the same as that of the tangent to the curve at that point. The direction usually changes from point to point, and the tangent line rotates as we move along the curve from a selected initial point.

If s is the distance of the point in question from the initial point, measured along the curve, and ϕ the angle the tangent at this point (s) makes with any fixed line in the plane, then

$$\phi \text{ is a function of } s, \quad \text{.................}(1)$$

and $\dfrac{d\phi}{ds}$ is the rate at which ϕ changes with s at the point (s) of the curve at which it is calculated. This derivative will in general vary from point to point.

Deviation.

It is convenient to have a word for the total change in direction which takes place as we go from one point on the curve to another. Thus, in fig. 1, if AQ, the tangent at A, is taken as the datum line, then ψ is the total change of direction between A and B, where ψ is the angle at which the tangent at B cuts

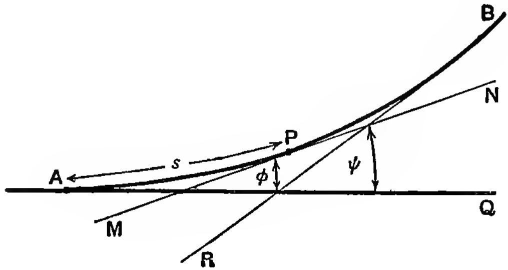


Fig. 1

AQ. We will call the total change of direction *the deviation*, then,

in fig. 1, the deviation from A to B is ψ.

Rate of Change of Deviation.

Suppose we measure the length of the curve itself from A (fig. 1). Select any point P between A and B, and draw the tangent at P (MN). The angle ϕ at which this line cuts AQ, the tangent at A, is the deviation between A and P.

Let s be the length of the curve AP, then $\dfrac{d\phi}{ds}$ measures the rate of change of ϕ with s at the point (s) (see p. 65), i.e. it measures the rate of change of deviation with the distance along the curve at P.

We will now seek a geometrical meaning of $d\phi/ds$ when the curve in question is a circle.

Curvature of a Circle.

Let ABC (fig. 2) be any given circle of radius r. Draw the tangent at A, AQ. Let O be the centre

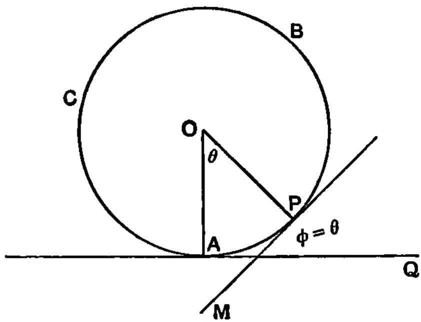


Fig. 2

of the circle. Join OA. Select any point P on the circle. Draw PM the tangent at P, cutting AQ at an angle ϕ. Join OP.

$$\text{Since PM} \perp \text{OP,}$$
$$\text{and AQ} \perp \text{OA,}$$

the angle between PM and AQ must be equal to the angle between OP and OA.

Let this angle be θ radians,

$$\text{then } \phi = \theta.$$

But, by (5), p. 102, if we measure the length of the arc of the circle from A, so that the arc AP is s, then

$$s = r\theta$$
$$= r\phi.$$
$$\therefore \phi = \left(\frac{1}{r}\right)s,$$

r being the *constant* radius of the circle.

$$\therefore \frac{d\phi}{ds} = \frac{1}{r}, \quad \text{.......................(2)}$$

i.e. $\frac{d\phi}{ds}$ is the reciprocal of the radius of the circle.

The *deviation per unit length of arc* $\left(\frac{d\phi}{ds}\right)$ is therefore constant for a circle, and equal to the reciprocal of the radius of the circle. This quantity—the deviation per unit length of arc—is called the *curvature*. If we define "curvature" in this way, the curvature of a circle is the same at any point on it, and this conclusion is in accord with the general meaning we attach to the word "curvature" when we use it in an ordinary way, with no thought of its precise mathematical definition.

Curvature of Plane Curves.

The geometrical meaning we have found for $d\phi/ds$ by considering a circle leads to a general definition of curvature which is applicable to any curve.

$d\phi/ds$ is the "curvature" of a plane curve at the point s.

Radius of Curvature.

The reciprocal of the curvature of a plane curve at any point on it is called the radius of curvature at that point.

It follows from equation (2) above, that the radius of curvature of a circle is simply its radius.

Both the curvature and the radius of curvature vary from point to point along the curve; it is only in the case of a special curve—a circle—that the curvature and the radius of curvature are constant from point to point.

Circle of Curvature.

Let A be any point on a curve CAB (fig. 3), PQ the tangent at A. Draw AN the normal at A. From A mark off AN, the radius of curvature. The point N is called the *centre of curvature*. With N as centre, describe a circle, radius NA. Then this

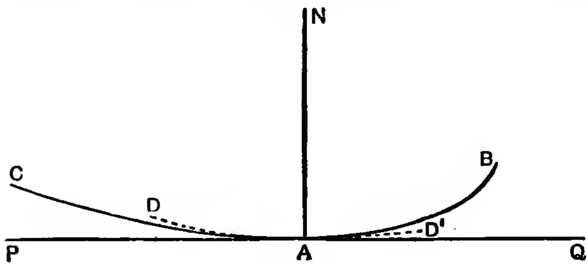


Fig. 3

circle and the curve AB have PQ for a common tangent, and the same curvature at A.

The circle is called the *circle of curvature* at A.

Usually, the circle of curvature cuts the curve at the point A, lying between the curve and the tangent on one side of A, and inside the curve on the other side of A.

That this is true, may be seen by considering a family of circles touching the curve at A, and therefore having their centres on AN. If a centre be taken between A and N, the curvature of the circle will be greater than the curvature of the curve at A, and the circle will therefore lie wholly inside the curve near A. If the centre of the circle be taken beyond N, the curvature of the circle will be less than the curvature of the curve at A, and hence the circle will lie wholly outside the curve near A.

If the centre of the circle is *at* N, the circle will be inside the curve on that side of A towards which the curvature of the curve decreases, but outside it on the other side of A.

At a point where the curvature is a maximum or a minimum, the circle of curvature does not cut the curve.

The reader is advised to verify these statements, with the help of a pair of compasses, on an actual diagram. He will soon be convinced of their truth, and will find that they give a good method for finding the centre of curvature.

Transformation of Curvature Formulæ.

$\frac{d\phi}{ds}$ is not readily calculated, as it stands, as we usually have the equation of the curve in x, y co-ordinates.

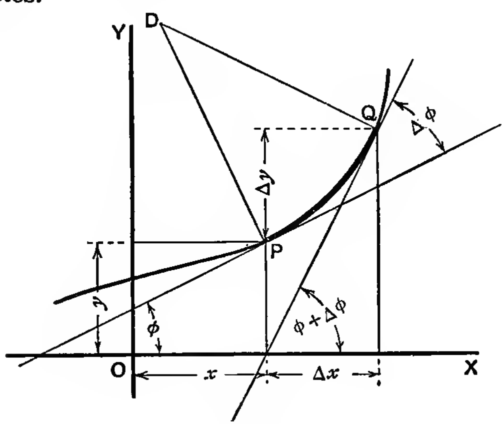


Fig. 4

To transform $d\phi/ds$ to x, y co-ordinates.

Choose axes as shown in fig. 4.

$$\text{Then } \tan\phi = \frac{dy}{dx},$$

$$\text{and } \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \text{ by p. 169}$$

$$= \sqrt{1 + \tan^2\phi}$$

$$= \sec\phi.$$

Also, by differentiating

$$\tan\phi = \frac{dy}{dx}, \text{ we get}$$

$$\sec^2\phi \frac{d\phi}{dx} = \frac{d^2y}{dx^2}.$$

$$\text{Now } \frac{d\phi}{dx} = \frac{d\phi}{ds}\frac{ds}{dx} = \frac{1}{\rho}\sec\phi,$$

when ρ is the radius of curvature at P.

$$\therefore \frac{1}{\rho}\sec^3\phi = \frac{d^2y}{dx^2}.$$

$$\text{Also } \sec^2\phi = 1 + \tan^2\phi$$

$$= 1 + \left(\frac{dy}{dx}\right)^2.$$

$$\therefore \sec^3\phi = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}.$$

$$\therefore \frac{1}{\rho} = \frac{d^2y}{dx^2} \Big/ \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}} \quad \ldots\ldots(3)$$

$$= \text{curvature at P } (x, y).$$

Sign.

The right-hand side of equation (3) may be taken with the positive or negative sign, as it involves the square root of $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^3$, which is always positive.

The correct sign to use can be easily seen from the shape of the graph, thus, in fig. 4, for instance, we have

$$\frac{1}{\rho} = \frac{d\phi}{ds},$$

and in going from P to Q the tangent rotates in the counter-clockwise direction, hence $\Delta\phi$ is positive, and, if we measure s in the direction from P to Q, Δs is positive, hence $\Delta\phi/\Delta s$ is positive and $\frac{1}{\rho}$ must be positive, so we take the positive sign with

$$\frac{d^2y}{dx^2}\Big/\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}.$$

WORKED EXAMPLES

1. Find the radius of curvature of a parabola $x^2 = 4ay$ at the vertex (0, 0).

Fig. 5 shows the graph of the given parabola.

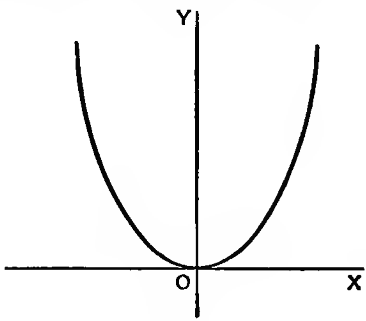


Fig. 5

$$y = \frac{x^2}{4a}.$$

$$\frac{dy}{dx} = \frac{x}{2a}.$$

$$\frac{d^2y}{dx^2} = \frac{1}{2a}.$$

$$\therefore \frac{1}{\rho} = \frac{\dfrac{d^2y}{dx^2}}{\left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}$$

$$= \frac{\dfrac{1}{2a}}{\left(1 + \dfrac{x^2}{4a^2}\right)^{\frac{3}{2}}}$$

$$= \frac{1}{2a}\Big/\frac{(4a^2 + x^2)^{\frac{3}{2}}}{8a^3}$$

$$= \frac{4a^2}{(4a^2 + x^2)^{\frac{3}{2}}}$$

$$= \frac{1}{2a} \text{ when } x = 0.$$

$$\therefore \rho = 2a \text{ when } x = 0.$$

Geometrical Treatment of Curvature.

Let PQ be a portion of a curve AB (fig. 6).

Let PD be the tangent at P, and QC the tangent at Q. Let C be the intersection of the tangents at P and Q.

Through P draw PO ⊥ PD, and through Q draw QO ⊥ QC, and let these lines meet in O. Let the tangent at P and the normal at Q (OQ produced) meet in D.

Then △OPD has a right angle at P.

Draw PE parallel to CQ, meeting OQ in E.

Let the angle POQ be $\Delta\phi$, and the *arc* PQ, Δs.

Let OP be denoted by r.

We will suppose ϕ, the deviation, is a continuous function of s, the length of the curve from A.

Then, no matter what pair of values, ϕ, s, we take

$$\Delta\phi \rightarrow 0 \text{ as } \Delta s \rightarrow 0.$$

It is evident from the figure that

$$PD > PC + CQ > \Delta s > \text{chord } PQ > PE.$$

$$\therefore\ PD > \Delta s > PE.$$

$$\therefore\ \frac{1}{PD} < \frac{1}{\Delta s} < \frac{1}{PE}.$$

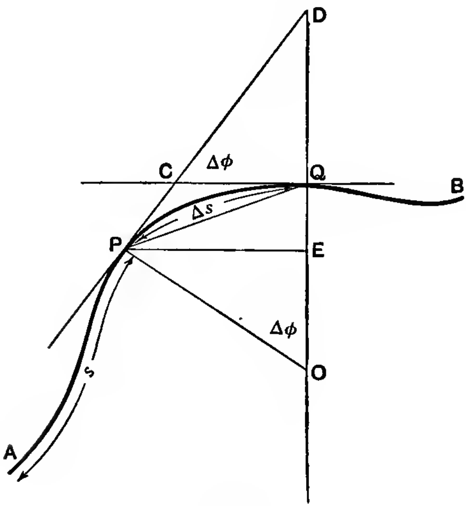


Fig. 6

But

$$PD = OP \tan \Delta\phi = r \tan \Delta\phi,$$

$$\text{and } PE = OP \sin \Delta\phi = r \sin \Delta\phi.$$

$$\therefore\ \frac{1}{r \tan \Delta\phi} < \frac{1}{\Delta s} < \frac{1}{r \sin \Delta\phi}.$$

$$\therefore\ \frac{\Delta\phi}{r \tan \Delta\phi} < \frac{\Delta\phi}{\Delta s} < \frac{\Delta\phi}{r \sin \Delta\phi}. \quad \ldots\ldots\ldots(a)$$

Now, let P remain fixed, and let Q move up towards P, so that $\Delta s \to 0$, then

$$\underset{\Delta s \to 0}{\mathrm{Lt}}\left[\frac{1}{r}\frac{\Delta\phi}{\tan\Delta\phi}\right] = \underset{\Delta s \to 0}{\mathrm{Lt}}\left(\frac{1}{r}\right) \times \underset{\Delta s \to 0}{\mathrm{Lt}}\left(\frac{\Delta\phi}{\tan\Delta\phi}\right),$$

and as $\Delta s \to 0$, $\Delta\phi \to 0$.

$$\therefore \underset{\Delta s \to 0}{\mathrm{Lt}}\left(\frac{\Delta\phi}{\tan\Delta\phi}\right) = \underset{\Delta\phi \to 0}{\mathrm{Lt}}\left(\frac{\Delta\phi}{\tan\Delta\phi}\right) = 1, \text{ by p. 106.}$$

$$\therefore \underset{\Delta s \to 0}{\mathrm{Lt}}\left[\frac{1}{r}\frac{\Delta\phi}{\tan\Delta\phi}\right] = \underset{\Delta s \to 0}{\mathrm{Lt}}\left(\frac{1}{r}\right).$$

Similarly,

$$\underset{\Delta s \to 0}{\mathrm{Lt}}\left[\frac{1}{r}\frac{\Delta\phi}{\sin\Delta\phi}\right] = \underset{\Delta s \to 0}{\mathrm{Lt}}\left(\frac{1}{r}\right)$$

by using the formula (11) on p. 106, and

$$\underset{\Delta s \to 0}{\mathrm{Lt}}\frac{\Delta\phi}{\Delta s} = \frac{d\phi}{ds}.$$

Hence, since the bounds in the inequality (*a*) both have the same limit, viz. $\underset{\Delta s \to 0}{\mathrm{Lt}}\left(\frac{1}{r}\right)$,

$$\therefore \frac{d\phi}{ds} = \underset{\Delta s \to 0}{\mathrm{Lt}}\left(\frac{1}{r}\right)$$

$$= \underset{\Delta s \to 0}{\mathrm{Lt}}\left(\frac{1}{\mathrm{OP}}\right) \quad \text{.............(4)}$$

I.e. $ds/d\phi$ is the value of the limiting length to which OP approaches, as the length of the arc, PQ, between the points at which the normals OP and OQ are drawn, shrinks towards zero.

This length is the radius of curvature of the curve at P, for we have defined the radius of curvature at P as the reciprocal of the curvature, $d\phi/ds$.

If Δs is sufficiently small we can write

$$\delta s = \rho \delta \phi, \quad \text{.....................(5)}$$

with sufficient accuracy for all practical purposes.

Total Length of a Curve.

The equation

$$\frac{ds}{d\phi} = \rho$$

enables us to express the total length of a curve between any two points A and B, as

$$S = \int_{\phi_0}^{\phi_1} \rho d\phi, \quad \text{.....................(6)}$$

where ϕ_0 is the deviation at A and ϕ at$_1$ B.

If we measure the angle ϕ with reference to the tangent at A, $\phi_0 = 0$.

$$\therefore S = \int_0^{\phi_1} \rho d\phi. \quad \text{.....................(7)}$$

Graphical Methods.

A graphical example will help to crystallize the conclusions so far reached.

Draw any smooth curve ACDB, about 12 in. long, on a sheet of drawing paper.

Step off points P_1, P_2, P_3, &c., with dividers set with the points $\frac{1}{2}$ in. apart. Draw the chords AP_1, P_1P_2, P_2P_3, &c.

If these chords are practically indistinguishable from the given curve, we can take the length of the curve between any pair of points as the length of the chord between the points.

Draw tangents at A, P_1, P_2, P_3, &c., and draw lines through A, P_1, P_2, P_3, &c., perpendicular to these tangents, each to each.

These tangents must, in general, be drawn by eye. With care, they can be drawn with fair accuracy.

These straight lines are the *normals* to the given curve at the respective points. Let the normals at A and P_1 intersect at Q_1. Measure P_1Q_1; then the

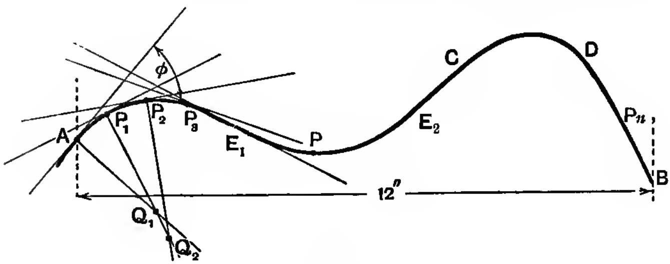


Fig. 7

length of P_1Q_1 is, approximately, the radius of curvature at P_1. Strike the arc AP_1P_2 with Q_1 as centre and radius P_1Q_1. This *circular arc* should be indistinguishable from the actual curve between A and P_2, if the length P_1Q_1 is a good approximation to the true radius of curvature at P_1.

Plot this value of ρ on a graph of ρ in terms of s; this is the value of ρ for s = chord AP_1, if we measure the length of the arc from A.

Repeat this process for each of the points P_2, P_3, P_4, . . . P_n, and plot the values of ρ_1, ρ_2, . . . ρ_n in terms of s. See fig. 8.

A curve giving the radius of curvature (ρ) for given values of s, between P_1 and P_n, is thus obtained.

In plotting this graph, meantime omit the values of ρ at the end points A and B, as these points require special treatment, since the curve *begins* at A and *ends* at B, and the curve does not have an *ordinary* tangent at these points.

This curve (fig. 8) shows clearly how the radius of curvature changes as we go along the curve. At certain values of s (OE_1, OE_2) the radius gets very

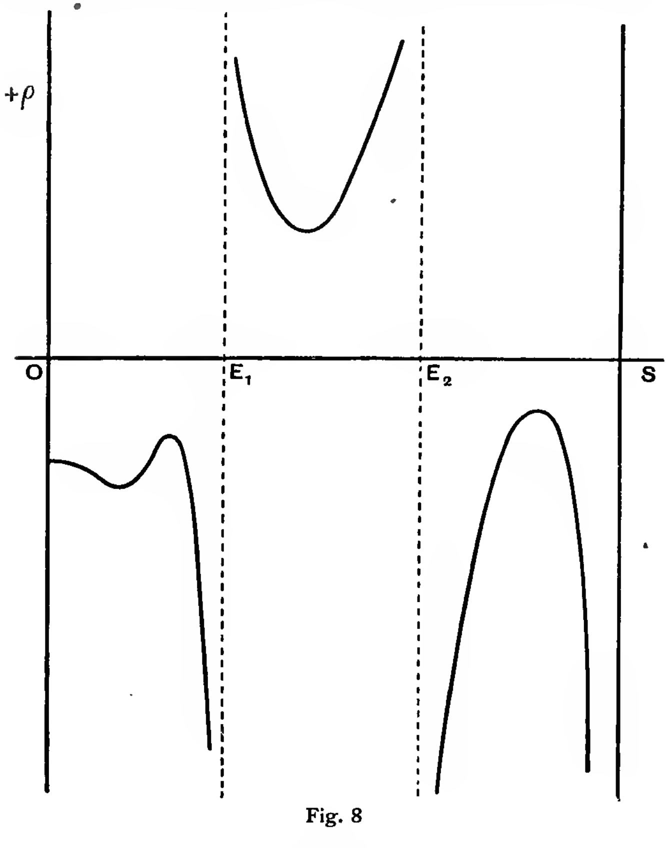


Fig. 8

great, and changes sign in the neighbourhood of these points. This means that as the corresponding points E_1, E_2 (fig. 7) are approached and passed, on the actual curve ACDB the deviation of the curve ceases to increase (decrease) and begins to decrease (increase). We can avoid these very great values of ρ by plotting, not ρ, but $\frac{1}{\rho}$, i.e. by plotting *curvature* and not radius of curvature.

The curvature can be obtained from fig. 7 directly.

Suppose $\Delta\phi_1$ is the angle between the tangents at P_1 and P_2.

Then the deviation in traversing the distance P_1P_2 is $\Delta\phi_1$, and the average curvature between P_1 and P_2 is

$$\left(\frac{\Delta\phi_1}{P_1P_2}\right) \text{ radians per unit length.}$$

If the step P_1P_2 is small enough, this average value may be taken as the value of the curvature at P_1.

If P_1P_2 is $\frac{1}{2}$ in., for instance, then the curvature at P_1 is approximately

$$\left(\frac{\Delta\phi_1}{\frac{1}{2}}\right) = 2\,\Delta\phi_1 \text{ radians per inch.}$$

The angle $(\Delta\phi_1)$ is easily measured in radians by striking the arc of a circle of radius 1 in. with the centre at the point of intersection of the tangents at P_1 and P_2. Measure the arc of this circle which is subtended by $\Delta\phi_1$; then, if δ_1 is the length of this arc in inches,

$$\Delta\phi_1 = \delta_1.$$

The *chord* corresponding to the arc δ_1 is sufficiently nearly equal to the arc δ_1, if the angle $\Delta\phi_1$ is sufficiently small, as it will be, if the steps P_1P_2, P_2P_3, &c., are sufficiently small.

The curvature at P is therefore given approximately by

$$\frac{1}{\rho} = (2 \times \text{chord } \delta_1) \text{ radians per inch,}$$

in this instance. If the steps P_1P_2, P_2P_3, &c., are all equal, the curvatures at successive points P_1, P_2, P_3, &c., are proportional to δ_1, δ_2, δ_3, &c.

In fig. 9, $\frac{1}{\rho}$ is plotted from fig. 8.

The work can now be checked by integration.

Let T be any point in the graph of $\frac{1}{\rho}$ in terms of s (P in fig. 7 corresponds to this point).

The area shaded, reckoned positive when above the axis and negative when below it, is

$$\int_{OP_1}^{OP} \frac{1}{\rho} ds.$$

But

$$\frac{d\phi}{ds} = \frac{1}{\rho}.$$

$$\therefore \int_{OP_1}^{OP} \frac{1}{\rho} ds = (\phi_P - \phi_{P_1}). \ldots\ldots\ldots\ldots\ldots (8)$$

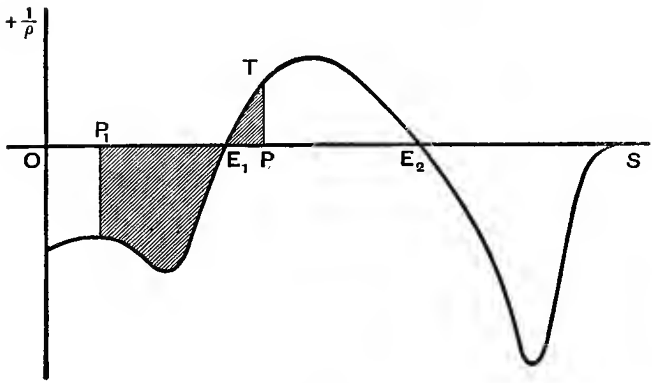


Fig. 9

If, then, the work is accurate, the area shaded should be algebraically equal to the total deviation between P_1 and P.

This method is, at best, only roughly accurate as it is difficult to draw the tangents at P_1, P_2, P_3, &c. (fig. 7), accurately with the eye as the only guide. See p. 68, where the same difficulty is referred to in connection with the graphical problem of determining instantaneous velocities from a distance-time curve. The method shows, however, the general way in which the curvature $\left(\frac{1}{\rho}\right)$ varies from point to point, and can be applied to *any* smooth curve, whether we know its mathematical equation or not. The *mathematical* methods already described cannot be applied till we know the algebraical equation to the curve, and even when we do, the finding of curvature, analytically, is often a tedious proceeding. But, as a compensation, the analytical method is theoretically accurate.

Physical Applications.

We shall conclude this chapter with some examples of physical applications of the preceding theory.

1. MOTION IN A CURVE.—(*a*) *Velocity*.

Suppose a particle P moves on a curved line AQ.

Let s denote the distance of the particle, measured along the curve from some selected point A, so that s is the length of the arc AP.

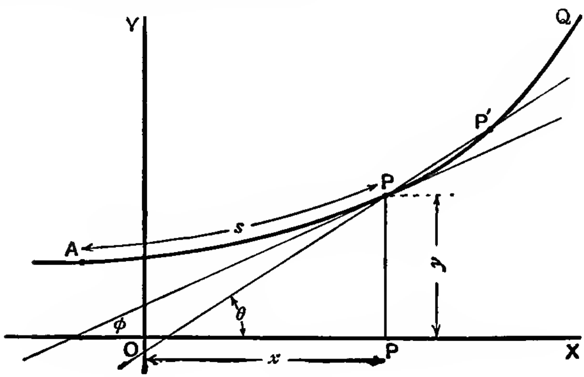


Fig. 10

Let the co-ordinates of P be x, y.

Suppose the particle moves from P to P′ in time Δt.

On p. 57 we discussed the *average velocity* of a rectilinear motion. In rectilinear motion there is no change in the direction of motion except such as can be dealt with by the *sign* of the speed (or magnitude) of the motion. The speed is positive when in one direction, and negative when in the only other possible direction—the *opposite* direction. In general, *the direction* of a velocity must be specified, as well as its magnitude, just as the direction of a displacement must be specified, as well as its amount. A *uniform* velocity must therefore be not only constant as regards its speed (magnitude), but must also maintain a constant direction. With this extended idea of *uniform* velocity, we can deal with the velocity of a particle on a curve.

The *average* velocity between P and P′ is evidently a velocity of magnitude

$$\left(\frac{\text{chord PP}'}{\Delta t}\right)$$

in the direction θ, the angle θ being the angle PP′ makes with the OX axis.

At P draw the tangent, cutting OX at the angle ϕ.

Now, let $\Delta t \longrightarrow 0$. As $\Delta t \longrightarrow 0$, the chord becomes infinitesimal, and the chord PP′ and the *arc* PP′ differ only by infinitesimals of higher order than the first, so we may put δs instead of PP′, when $\Delta t \longrightarrow 0$. Further, as $\Delta t \longrightarrow 0$, $\theta \longrightarrow \phi$, the direction of the tangent at P. Hence

$$\underset{\Delta t \longrightarrow 0}{\text{Lt}} \left(\frac{\text{PP}'}{\Delta t}\right) = \frac{ds}{dt},$$

$$\text{and} \quad \underset{\Delta t \longrightarrow 0}{\text{Lt}} (\theta) = \phi,$$

and these limits *define* the instantaneous velocity of the particle.

The instantaneous velocity of the particle at P is *ds*/*dt*, and its direction is that of the tangent at P.

(*b*) *Acceleration.*

Choose the tangent and normal at P as axes, as shown in fig. 11.

At P, let the velocity be v along OX, and at P′, $(v + \Delta v)$ along TP′, the tangent at P′.

Tangential Acceleration.

The component of $(v + \Delta v)$ *along OX* is $(v + \Delta v)\cos\Delta\phi$. Hence the acceleration at P, along the tangent at P, (f_t), is given by

$$f_t = \underset{\Delta t \to 0}{\text{Lt}} \left[\frac{(v+\Delta v)\cos\Delta\phi - v}{\Delta t}\right].$$

Now, $1 > \cos\Delta\phi > 1 - \frac{1}{2}(\Delta\phi)^2$, see p. 105.

$\therefore$ we can put $\cos\Delta\phi = 1 - \frac{1}{2}\kappa(\Delta\phi)^2$,

where κ is a positive fraction.

$$\therefore \frac{(v+\Delta v)\cos\Delta\phi - v}{\Delta t} = \frac{(v+\Delta v)\left(1-\frac{1}{2}\kappa(\Delta\phi)^2\right) - v}{\Delta t}$$

$$= \frac{v+\Delta v - \frac{1}{2}\kappa v(\Delta\phi)^2 - \frac{1}{2}\kappa\Delta v(\Delta\phi)^2 - v}{\Delta t}.$$

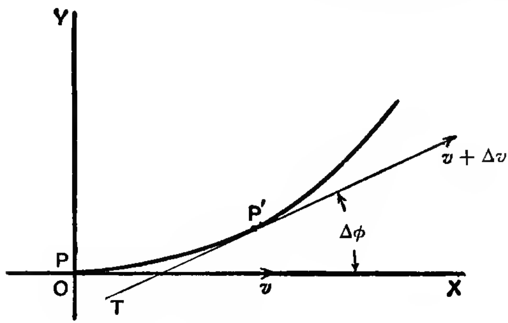


Fig. 11

If we retain only infinitesimals of the first order, this expression gives

$$\frac{(v+\Delta v)\cos\Delta\phi - v}{\Delta t} = \frac{\Delta v}{\Delta t}.$$

$$\therefore \underset{\Delta t \to 0}{\text{Lt}} \left[\frac{(v+\Delta v)\cos\Delta\phi - v}{\Delta t}\right] = \frac{dv}{dt},$$

$$\text{and since } v = \frac{ds}{dt},$$

$$\frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

Hence the acceleration along the tangent at P is $\frac{dv}{dt}$ or $\frac{d^2s}{dt^2}$.

Normal Acceleration.

Now, resolve $(v + \Delta v)$ along the normal at P.

The normal component is $(v + \Delta v) \sin \Delta\phi$. Hence the normal acceleration at P is

$$f_n = \underset{\Delta t \to 0}{\mathrm{Lt}} \left[\frac{(v + \Delta v) \sin \Delta\phi}{\Delta t}\right]$$

$$= \underset{\Delta t \to 0}{\mathrm{Lt}} \left[\frac{v + \Delta v}{v}\, v\, \frac{\sin \Delta\phi}{\Delta\phi}\, \frac{\Delta\phi}{\Delta s}\, \frac{\Delta s}{\Delta t}\right].$$

Taking the limit of each of these four products in turn, we get

$$\underset{\Delta t \to 0}{\mathrm{Lt}} \left(\frac{v + \Delta v}{v}\right) = 1,$$

$$\underset{\Delta t \to 0}{\mathrm{Lt}} (v) = v,$$

$$\underset{\Delta t \to 0}{\mathrm{Lt}} \left(\frac{\sin \Delta\phi}{\Delta\phi}\right) = 1,$$

$$\underset{\Delta t \to 0}{\mathrm{Lt}} \left(\frac{\Delta\phi}{\Delta s}\right) = \frac{1}{\rho}, \text{ the curvature at P,}$$

$$\text{and} \quad \underset{\Delta t \to 0}{\mathrm{Lt}} \left(\frac{\Delta s}{\Delta t}\right) = \frac{ds}{dt} = v.$$

$$\therefore f_n = 1 \times v \times 1 \times \frac{1}{\rho} \times v = \frac{v^2}{\rho}.$$

Hence the normal acceleration is the square of the speed divided by the radius of curvature.

The results are:

$$f_t = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

$$f_n = \frac{v^2}{\rho}.$$

There is therefore *always* a normal acceleration when a particle moves along a curve. There is also a tangential acceleration if its speed changes from point to point.

The normal acceleration arises from the curvature of the path, and therefore from the change of direction of the motion; the tangential acceleration from the change in speed.

Example.—A particle moves at a uniform speed round an ellipse. Find how the normal acceleration varies from point to point.

Let the centre of the ellipse be the origin, the major axis the OX axis, and the minor axis the OY axis; then (see fig. 4, p. 9) the equation of the ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1, \quad \text{.........................(9)}$$

where a is the semi-major axis and b the semi-minor axis.

Equation (9) gives

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}.$$

Differentiating both sides with regard to x (y being a function of x), we get

$$\frac{2y}{b^2}\frac{dy}{dx} = -\frac{2x}{a^2}.$$

$$\therefore \frac{dy}{dx} = -\frac{b^2}{a^2}\frac{x}{y},$$

$$\text{and } \frac{d^2y}{dx^2} = -\frac{b^2}{a^2}\frac{1}{y} - \frac{b^2x}{a^2}\left(-\frac{1}{y^2}\frac{dy}{dx}\right)$$

$$= -\frac{b^2}{a^2y} - \frac{b^2x}{a^2}\left(-\frac{1}{y^2} \times -\frac{b^2}{a^2}\frac{x}{y}\right)$$

$$= -\left[\frac{b^2}{a^2y} + \frac{b^4x^2}{a^4y^3}\right]$$

$$= -\frac{b^2}{a^2y}\left[1 + \frac{b^2x^2}{a^2y^2}\right]$$

$$= -\frac{b^2}{a^2y}\left[\frac{a^2y^2 + b^2x^2}{a^2y^2}\right].$$

$$a^2y^2 + b^2x^2 = a^2b^2 \text{ by equation (9).}$$

$$\therefore \frac{d^2y}{dx^2} = -\frac{b^2}{a^2y}\frac{a^2b^2}{a^2y^2}$$

$$= -\frac{b^4}{a^2y^3}.$$

$$\text{Now, } \frac{1}{\rho} = \frac{\dfrac{d^2y}{dx^2}}{\left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}} \text{ by equation (3), p. 194.}$$

$$\therefore \frac{1}{\rho} = \frac{-\dfrac{b^4}{a^2y^3}}{\left\{1 + \dfrac{b^4x^2}{a^4y^2}\right\}^{\frac{3}{2}}}$$

$$= -\frac{a^4b^4}{(a^4y^2 + b^4x^2)^{\frac{3}{2}}}.$$

If the speed of the particle is v, we get

$$f_n = \frac{v^2a^4b^4}{(a^4y^2 + b^4x^2)^{\frac{3}{2}}}, \qquad\text{.........................(10)}$$

and this acceleration is directed towards the interior of the ellipse.

At the end of the major axis $(a, 0)$

$$f_1 = \frac{v^2a}{b^2},$$

and at the end of the minor axis

$$f_2 = \frac{v^2b}{a^2}.$$

The normal acceleration anywhere else (x, y) is given by equation (10).

The tangential acceleration is zero as $\dfrac{ds}{dt}$ is constant, by hypothesis.

2. The Bending of Beams.

Let AB be an I girder bent in one plane by forces in that plane, as shown in fig. 12 (a), and supported by knife edges at A and B. Choose rectangular axes as shown in the figure.

Let OO″O′ be the line passing through the centroid of each cross-section of the girder. This line is called the *axis* or centre line of the beam.

Suppose the cross-section of the girder is as shown in fig. 12 (*b*). When the girder "hogs", filaments of material originally parallel to the axis (supposed

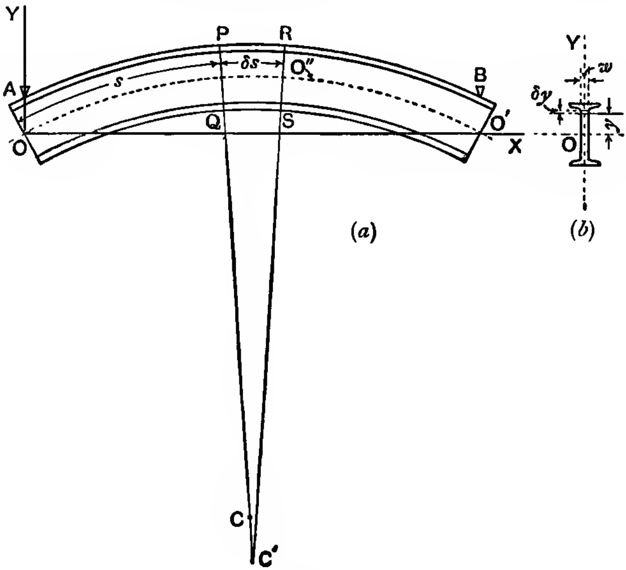


Fig. 12

initially straight) stretch if they are above the axis and shrink if they are below it, provided the axis itself does not stretch nor shrink.

The portion of the beam to the left of PQ is held in equilibrium by the applied system of forces acting on that portion, plus the stresses acting in the cross-section PQ, arising from the other portion of the beam.

These stresses can be reduced to

(1) a force T, tangential to the axis;
(2) a shearing force N

—these forces acting at the centroid of the cross-section; and

(3) a couple M.

These three quantities can be found by considering the statics of the portion of the beam on the left of PQ. The stress system at any cross-section, distant s from O (measured along the axis), is therefore determinate.

One of the problems we often wish to solve is to find the equation of the deflected axis. This equation will clearly depend on the dimensions and the elastic constants of the beam. To determine it we have to express the couple M—the bending-moment, as it is called—in terms of the dimensions of the beam and the elastic constants. We shall suppose that T is so small that the axis itself does not change sensibly in length.

Let C be the centre of curvature of the axis at P.

Consider a small length of the beam δs (PQSR), and let us calculate what happens to the filaments of material distant y from an axis through the centroid of the section and perpendicular to the plane of bending (fig. 12 (*b*)). Let δS be the cross-sectional area of the filament at y.

The original length of this filament is δs; the stretched length is $(\rho + y)\delta\phi$, ignoring infinitesimals of higher order than the first.

But $\delta s = \rho\delta\phi$ to the same order of accuracy. Hence the strain of the filament is

$$\frac{(\rho + y)\delta\phi - \rho\delta\phi}{\rho\delta\phi} = \frac{y}{\rho}.$$

If E is Young's Modulus, the stress in this element is $\frac{\mathrm{E}y}{\rho}$. Hence the tension in it is $\frac{\mathrm{E}y}{\rho}\delta\mathrm{S}$.

Taking moments about an axis perpendicular to the plane of bending and passing through the centroid of the section PQ, we see that the contribution this filament makes to the total couple is

$$\frac{Ey^2\delta S}{\rho}.$$

The whole couple is therefore

$$\int\frac{Ey^2dS}{\rho} = \int_{-\frac{d}{2}}^{+\frac{d}{2}}\frac{Ey^2wdy}{\rho},$$

where w is the thickness of the beam at y (fig. 12 (b)).

$$\therefore\ G = \frac{E}{\rho}\int_{-\frac{d}{2}}^{+\frac{d}{2}} wy^2dy,$$

where G is the total couple.

But $\int_{-\frac{d}{2}}^{+\frac{d}{2}} wy^2dy$ is the moment of inertia of the cross-section, assuming it to be of unit surface density.

Calling this quantity I, we get

$$G = \frac{EI}{\rho}.$$

This couple must be numerically equal to M.

$$\therefore\ \frac{EI}{\rho} = \pm M,$$

$$\text{i.e. } EI\frac{\dfrac{d^2y}{dx^2}}{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}} = \pm M.$$

If the deflection is never very great, dy/dx is small relative to unity, and much more is $(dy/dx)^2$. Hence we can neglect $(dy/dx)^2$ and put

$$\mathrm{EI}\frac{d^2y}{dx^2} = \pm \mathrm{M}.$$

Sign.

The sign to be given to the right-hand side of the equation

$$\mathrm{EI}\frac{d^2y}{dx^2} = \pm \mathrm{M}$$

is settled by a convention.

If the beam bends as shown in the figure, the bending-moment, M, arising from the portion of the beam on the *right* of PQ is clearly clockwise, i.e. *negative*, with the usual convention that counter-clockwise couples are positive.

But $\frac{d^2y}{dx^2}$ is negative too, since $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$, and the inclination of the axis of the beam $\left(\frac{dy}{dx}\right)$ *decreases* as we proceed along the beam in the direction of x increasing; thus $\mathrm{EI}\frac{d^2y}{dx^2}$ and M have the same sign, i.e.

$$\mathrm{EI}\frac{d^2y}{dx^2} = \mathrm{M}.$$

This is the equation that gives the deflected axis of the beam.

The above is the Bernoulli-Euler theory of the bending of beams. A state of strain is assumed defined by the properties:

1. The axis is unaltered in length.
2. The cross-sections remain plane, undistorted, and perpendicular to the axis.
3. There is no extension or contraction of fibres, perpendicular to their length.

The mathematical theory of elasticity shows that the actual state of strain, even under simple distributions of load, does not exactly fulfil the above conditions.

In calculating the moment of resistance of the beam arising from these strains, we further suppose that the straining of the longitudinal filaments is not resisted by shearing stresses along the sides of them; in other words, we suppose the beam to be equivalent to a bundle of free wires placed parallel to the axis.

The exact theory shows that when any linear dimension of the cross-section is not more than, say, $\frac{1}{10}$ of the length of the beam, the formula for the deflection obtained on the simple theory given above is sufficiently correct.

Example.—A uniformly loaded beam, supported at both ends, carries a load of P lb. per foot run. Find the maximum deflection.

Let l be the length of the beam (fig. 13), OX, OY, the axes chosen in the usual way, where OX coincides with the axis of the *unbent* beam.

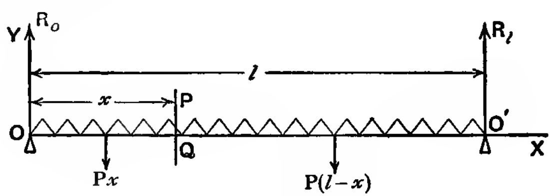


Fig. 13

The forces keeping the beam at rest are the loads acting vertically downwards, plus the reactions at O and O′.

Symmetry.—From the symmetry of the figure we see that:

(*a*) The reactions at O and O′ must be equal, each being $\frac{1}{2}Pl$, and they must act vertically upwards.

(*b*) The maximum deflection takes place at the mid-point of the beam, and is negative (downwards).

(*c*) At the mid-point (where the deflection is a maximum)

$$\frac{dy}{dx} = 0.$$

Boundary Conditions.—It is also clear that $y = 0$ when $x = 0$, or when $x = l$.

Consider any section PQ at x. The moments of the forces to the right of this section are, in the counter-clockwise direction,

$$R_l(l-x) - \frac{P(l-x)^2}{2}.$$

$$\therefore M = [\tfrac{1}{2}Pl(l-x) - \tfrac{1}{2}P(l-x)^2].$$

$$\therefore EI\frac{d^2y}{dx^2} = \tfrac{1}{2}P[l(l-x) - (l-x)^2].$$

Integrate this equation, and we get

$$EI\frac{dy}{dx} = \tfrac{1}{2}P\left[-\frac{l(l-x)^2}{2} + \frac{(l-x)^3}{3}\right] + C,$$

where C is a constant.

We can determine C by noting that

$$\frac{dy}{dx} = 0 \text{ at the mid-point, i.e. when } x = \frac{l}{2}.$$

$$\therefore 0 = \tfrac{1}{2}P\left[-\frac{l^3}{8} + \frac{l^3}{24}\right] + C.$$

$$\therefore C = \frac{Pl^3}{24}.$$

$$\therefore EI\frac{dy}{dx} = \tfrac{1}{2}P\left[-\frac{l(l-x)^2}{2} + \frac{(l-x)^3}{3}\right] + \frac{Pl^3}{24}.$$

Integrating this equation again, we get

$$EIy = \tfrac{1}{2}P\left[\frac{l(l-x)^3}{6} - \frac{(l-x)^4}{12}\right] + \frac{Pl^3}{24}x + D,$$

where D is a constant.

To determine D we use the fact that $y = 0$ when $x = 0$.

$$\therefore 0 = \tfrac{1}{2}P\left[\frac{l^4}{6} - \frac{l^4}{12}\right] + D.$$

$$D = -\frac{Pl^4}{24}.$$

$$\therefore EIy = \tfrac{1}{2}P\left[\frac{l(l-x)^3}{6} - \frac{(l-x)^4}{12}\right] - \frac{Pl^3}{24}(l-x).$$

Let δ be the maximum deflection, which is the value of y corresponding to $x = \frac{l}{2}$.

$$\therefore \; EI\delta = \tfrac{1}{2}P\left[\frac{l^4}{48} - \frac{l^4}{192}\right] - \frac{Pl^4}{48}$$

$$= -\frac{5}{384} Pl^4.$$

If W is the total load carried,

$$W = Pl,$$

$$\text{and } \delta = -\frac{5}{384}\frac{Wl^3}{EI}.$$

The negative sign shows that the beam is sagging.

Exercise 11

1. In the cubical parabola, where

$$a^2y = x^3,$$

show that the radius of curvature is

$$\frac{(a^4 + 9x^4)^{\frac{3}{2}}}{6a^4x}.$$

2. In the semi-cubical parabola,

$$ay^2 = x^3,$$

show that the radius of curvature is

$$\frac{(4a + 9x)^{\frac{3}{2}}}{6a}\sqrt{x}.$$

3. Show that the radius of curvature of the curve

$$y = A_0 + A_1x + A_2x^2 + \ldots A_nx^n,$$

n finite, where the A's are constants, at $x = 0$ is

$$\frac{[1 + A_1^2]^{\frac{3}{2}}}{2A_2}.$$

4. Find the radius of curvature of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

at the end of the major axis.

5. Show that if a particle of mass m lb. is attached to the end of a string and swings round in a circle of radius a ft., at constant angular velocity ω radians per second, the tension in the string is $\frac{m\omega^2 a}{g}$ lb., where g is the acceleration due to gravity (32·2 ft.-sec.^{2}).

6. The equations of a cycloid referred to the vertex as origin, and the tangent and normal at the vertex as axes OX, OY respectively, are

$$x = a(\theta + \sin\theta),$$
$$y = a(1 - \cos\theta),$$

where θ is the angle turned through by the generating circle of radius a.

Assuming these equations, show that the inclination of the tangent (ψ) to the OX axis at any point (x, y) on the cycloid is given by

$$\tan\psi = \tan\frac{\theta}{2}.$$

Hence show that the arc of the cyloid (s), measured from the vertex, is given by

$$s = 4a\sin\psi$$
$$= \sqrt{8ay}.$$

Show that the radius of curvature at P is given by

$$\rho = 4a\sqrt{1 - \frac{y}{2a}},$$

which is twice the distance of P from the base of the cycloid, when measured along the normal at P.

[The *base* is the line *uv* in fig. 3, p. 169.]

7. A particle of mass m oscillates under gravity in a cup. The cup is formed by the revolution of a cycloid about the normal at its vertex. Show that if the particle starts from rest at a distance A (measured on the surface of the cup) from the lowest point, the particle oscillates according to the equation

$$s = A \cos \omega t,$$

where ω is a constant.

Show that ω is equal to $\sqrt{(g/4a)}$.

Find the reaction on the cup at any point, using the results of the previous example.

8. A straight beam of uniform moment of inertia and negligible weight is bent by a couple applied at each end, the couples being coplanar.

Show that the centre line deflects into the arc of a circle.

Find an approximate equation for the deflected axis in the usual rectangular co-ordinates.

State generally how the shape of the curve is modified, if the weight of the beam is not negligible.

9. A uniform beam is bent in one plane by a single load w placed at its centre.

Find the approximate equation of the deflected axis.

What is the central deflection?

10. A straight wire of circular cross-section is clamped horizontally in a vice. A load w is applied at the free end.

Assuming this to be the only load to be considered, find the approximate equation of the deflected axis and the deflection at the end.

CHAPTER XII

Maxima and Minima

> "And so, from hour to hour, we ripe and ripe,
> And then, from hour to hour, we rot and rot;
> And thereby hangs a tale."—*As You Like It.*

The graphs of certain functions of x, say $\sin x$, rise and fall as we proceed along them in the direction OX, and exhibit points at which the value of the function is greater than, or less than, its value at neighbouring points. Such values of the function are called "turning values".

Turning Values of Functions.

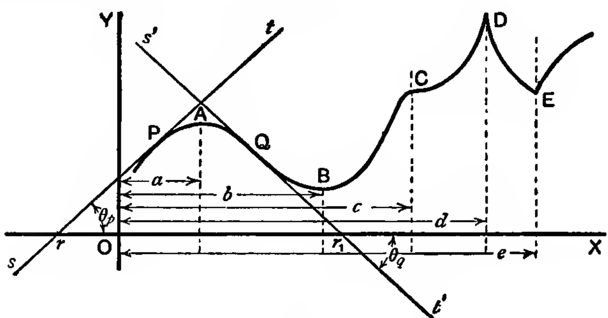


Fig. 1

Let ABCDE (fig. 1) be the graph of a continuous function of x, say $f(x)$;

$$\text{then } y = f(x). \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

At the point A corresponding to $x = a$, the ordinate y ceases to increase, and begins to decrease immediately afterwards, as we go along the graph in the direction of "x increasing", i.e. the positive direction. At B, the ordinate ceases to decrease, and begins to increase thereafter. The ordinate steadily increases in going through the point C. At D, we have the same state of things as at A, so far as decrease or increase of y is concerned.

Points such as A, B, and D are "turning points", and the values of y at these points are called "turning values" of the function. Points such as A and D are called *maximum* points, those such as B, *minimum* points.

These terms are used in a relative sense only; for instance, if we say that A is a maximum point, we do not mean that y is an absolute maximum at this point, for it is greater at D. What we do mean is that at A we can choose an interval within which the value $x = a$ is included, such that for any value of x (other than a) lying within this interval, the value $f(x)$ is less than the value $f(a)$. We must clearly exclude the value $x = a$, as at this value $f(x)$ of course equals $f(a)$.

Condition for a Maximum Point.

We can put these ideas very simply into symbols. If y is a maximum at $x = a$, we can choose a positive number δ so that

$$\left.\begin{array}{r} f(x) < f(a) \\ \text{so long as } 0 < |x - a| < \delta \end{array}\right\}. \qquad \text{............(2)}$$

The meaning of this definition is seen from fig. 2.

Condition for a Minimum Point.

In this condition we merely have to change the inequality sign in equation (2), thus

$$\left.\begin{array}{r} f(x) > f(a) \\ \text{so long as } 0 < |x - a| < \delta \end{array}\right\}, \quad \text{............(3)}$$

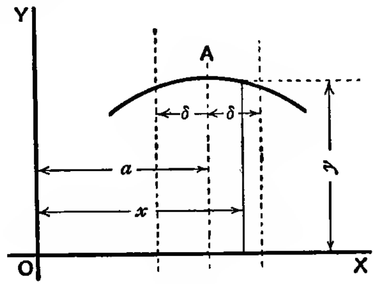


Fig. 2

where δ is a selected positive number, if y is a minimum when $x = a$.

Differentiable Functions.

It is evident that these tests apply just as much at points such as D and E as at A and B (see fig. 1). They therefore do not depend on the function having a definite finite derivative at a maximum or minimum point. In fact, these tests do not depend on the ideas of the Calculus at all.

The tests have the disadvantage that they cannot be applied until we select the value ($x = a$) at which we expect a turning value to exist. The tests themselves give no hint of these values. We can easily find a rule which will suggest values of x at which $f(x)$ has turning values, provided $f(x)$ is differentiable at every point within the range of x we are considering.

From fig. 1 it is clear that another property of $f(x)$, at a turning point of the type shown at A and B, is that the tangent to the graph is parallel to OX at points corresponding to such turning values, i.e. $\frac{dy}{dx} = 0$ at such maximum or minimum values of $f(x)$.

The reader should note that $\frac{dy}{dx}$ may equal zero at points which are neither maximum nor minimum points, e.g. at the point C (fig. 1). He should also observe what takes place at points like D and E. At the maximum point D the two branches have the same tangent, the gradient of which (tan 90°) is infinite; at the minimum point E the two branches have different tangents. Both at D and E, therefore, the gradient of the tangent is discontinuous. The rules now to be given will enable us to find all turning points, such as A and B. If maximum and minimum points like D or E occur, which is extremely unusual, they have to be found by special methods.

EXAMPLE *I*

Find the turning value of $x^2(x-1)$ between $x = 0$ and $x = 1$, excluding these values.

Method a.

$$\text{If } y = x^2(x-1) = f(x),$$

$$\frac{dy}{dx} = 3x^2 - 2x.$$

Put $3x^2 - 2x = 0$, whence $x = 0$ or $x = \frac{2}{3}$, i.e. between $x = 0$ and $x = 1$, $\frac{dy}{dx} = 0$ when $x = \frac{2}{3}$.

The value $x = \frac{2}{3}$ may therefore correspond to a turning value of $x^2(x-1)$.

To test this,

$$f(\tfrac{2}{3} + \delta) = (\tfrac{2}{3} + \delta)^2(\tfrac{2}{3} + \delta - 1)$$

$$= \delta^2 + \delta^3 - \tfrac{4}{27},$$

$$f(\tfrac{2}{3}) = -\tfrac{4}{27}.$$

$$\therefore f(\tfrac{2}{3} + \delta) - f(\tfrac{2}{3}) = \delta^2 + \delta^3,$$

and $\delta^2 + \delta^3$ is positive for all values of $\delta > -1$, i.e. so long as

$$0 < |x - \tfrac{2}{3}| < 1,$$
$$f(x) > f(\tfrac{2}{3}).$$

We have therefore found an interval including $x = \frac{2}{3}$, such that

$$f(x) > f(\tfrac{2}{3}) \text{ in this interval of } x,$$

i.e. $f(x)$ is a *minimum* at $x = \frac{2}{3}$.

Method b.—We can arrive at this result by another method.

1. $y = 0$, when $x = 0$ and $x = 1$.
2. Let $x = \kappa$, where κ is a positive fraction, then

$$f(\kappa) = \kappa^2(\kappa - 1),$$

κ^2 is positive and $(\kappa - 1)$ is negative. Hence $f(\kappa)$ is negative, i.e. $f(x)$ is negative for all values of x between 0 and 1.

3. There is *one* value only of x between 0 and 1 which makes dy/dx equal to zero, i.e. the tangent to the graph is parallel to OX at $x = \frac{2}{3}$ only.

4. On the other hand, there must be one value of x, say ξ, at least at which $x^2(x - 1)$ is a *minimum* between $x = 0$ and $x = 1$, since $x^2(x - 1)$ is zero at $x = 0$ and $x = 1$, and negative between these values of x.

5. Consequently this value of x must be that at which $\dfrac{dy}{dx} = 0$, i.e. $\xi = \frac{2}{3}$.

$$\therefore\ x^2(x - 1) \text{ is a minimum at } x = \tfrac{2}{3}.$$

Another Form of the Maximum-Minimum Test.

Consider the tangent at P (fig. 1), and trace what happens to it as the point of contact P moves along the curve through A to Q. The tangent line evidently rotates in a clockwise direction, becoming parallel to OX at A. Clockwise rotation means a *decrease* in the angle of inclination θ; hence $\tan\theta$ must decrease as P moves through A to Q, i.e. $\dfrac{dy}{dx}$ must decrease steadily between P and Q. θ is clearly positive at P, zero at A, and negative at Q; hence

At a maximum point, $\frac{dy}{dx}$ changes sign from positive to negative, as x increases through its critical value,

i.e. if $f(x)$ is a maximum when $x = a$, then $\frac{dy}{dx}$ is positive when $x < a$ and negative when $x > a$.

Similarly, at B, we see that:

At a minimum point, $\frac{dy}{dx}$ changes sign from negative to positive, as x increases through its critical value.

EXAMPLE *II*

$$\text{Let } y = (x - 1)(x - 2)^2.$$

$$\therefore \frac{dy}{dx} = (x - 2)(3x - 4),$$

which is zero when $x = 2$ and when $x = \frac{4}{3}$.

We are therefore led to consider these values of x as possible maximum or minimum points.

$$\text{When } \tfrac{4}{3} < x < 2,$$
$$(x - 2)(3x - 4) \text{ is negative.}$$

$$\text{When } x > 2,$$
$$(x - 2)(3x - 4) \text{ is positive;}$$

hence at $x = 2$, $\frac{dy}{dx}$ changes from negative to positive as x increases. y is therefore a *minimum* at $x = 2$.

Similarly, when $x < \frac{4}{3}$, $(x - 2)(3x - 4)$ is positive, and when $\frac{4}{3} < x < 2$, $(x - 2)(3x - 4)$ is negative, whence dy/dx changes from positive to negative as x increases through the value $x = \frac{4}{3}$.
$\therefore y$ is a maximum at $x = \frac{4}{3}$.

The reader should draw the graph of

$$y = (x - 1)(x - 2)^2,$$

and confirm these conclusions.

Analytical Test of Maxima and Minima.

There is yet another way in which this test may be put.

We have already seen that $\frac{dy}{dx} = 0$ at a turning point.

Suppose that at the turning point ($x = a$),

$$\frac{d^2y}{dx^2} \text{ is positive.}$$

Then $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ is positive, and when the derivative of a function is positive, for $x = a$ say, the function itself *increases* as x increases, through a.

Therefore $\frac{dy}{dx}$ must increase as x increases from $a - h$ to $a + h$, provided h is sufficiently small.

But $\frac{dy}{dx}$ is zero when $x = a$, hence $\frac{dy}{dx}$ must be negative when x is just less than a, and positive when x is just greater than a, i.e. $\frac{dy}{dx}$ changes sign from negative to positive as x increases through a, and hence y is a minimum.

The rule is, therefore:

If $dy/dx = 0$ at $x = a$, and d^2y/dx^2 is positive at $x = a$, then y is a minimum at $x = a$.

Similarly:

If $dy/dx = 0$ at $x = a$, and d^2y/dx^2 is negative at $x = a$, then y is a maximum at $x = a$.

If d^2y/dx^2 is zero when $x = a$, this test fails, and we then have to revert to the general tests already given.

EXAMPLE *III*

Let $y = x^2 - 15x + 3$,

then $\frac{dy}{dx} = 2x - 15$, which equals zero when $x = 7\frac{1}{2}$,

$\frac{d^2y}{dx^2} = 2$, which is always positive.

$\therefore$ the minimum value of $x^2 - 15x + 3$ occurs at $x = 7\frac{1}{2}$.

EXAMPLE *IV*

$$\text{Let } y = x^3,$$

$$\frac{dy}{dx} = 3x^2,$$

$$\text{and } \frac{d^2y}{dx^2} = 6x.$$

Both derivatives are zero at $x = 0$, and the analytical test fails. But we can see from the following considerations that there is no maximum nor minimum. It is evident that x^3 steadily increases as x increases from zero, and steadily decreases as x decreases from zero. x^3 has therefore no maximum nor minimum value for any value of x.

EXAMPLE *V*

Find the fraction which exceeds its square by the greatest amount. Let x be the required fraction; then $y = x - x^2$ must be a maximum.

This occurs when $x = \frac{1}{2}$.

We shall now discuss some problems in maxima and minima which arise in physics and engineering.

EXAMPLE *VI*

In an electrical transformer the losses of energy arise in the heating of the copper conductors (the "copper losses") and of the iron of the core (the "core losses"). Assuming that the total copper losses are proportional to the square of the secondary current, and that the core losses are constant at all currents, show that the transformer works at its maximum efficiency when the core loss is equal to the copper loss, assuming the power-factor of the load to be constant.

[The power-factor of an alternating-current circuit is the factor by which the product of pressure and current must be multiplied to get the power taken by the circuit; for instance, if a coil of wire carries an alternating current, I amperes as read by an ammeter, and if the pressure across the coil is E volts as read by a voltmeter, the power consumed by the circuit is not EI watts, but

$$\text{EI}\cos\phi, \text{ where } \cos\phi \text{ is the ``power-factor''}.$$

Its magnitude depends on the nature of the circuit.]

Let E stand for the r.m.s. secondary pressure,
I ,, ,, ,, ,, current,
cosϕ ,, ,, power-factor;

then EI cosϕ is the power delivered by the transformer.

Let W stand for the core loss (a constant), then the losses in the transformer are

$$\text{W} + a\text{I}^2, \text{ where } a \text{ is a constant;}$$

hence the power input to the transformer is

$$\text{EI}\cos\phi + \text{W} + a\text{I}^2;$$

hence the efficiency (η) is given by

$$\eta = \frac{\text{EI}\cos\phi}{\text{EI}\cos\phi + \text{W} + a\text{I}^2},$$

$$\frac{1}{\eta} = 1 + \frac{1}{\text{E}\cos\phi}[\text{WI}^{-1} + a\text{I}],$$

$$\frac{d}{d\text{I}}\left(\frac{1}{\eta}\right) = \frac{1}{\text{E}\cos\phi}[-\text{WI}^{-2} + a],$$

$$\text{and} \quad \frac{d^2}{d\text{I}^2}\left(\frac{1}{\eta}\right) = \frac{1}{\text{E}\cos\phi}[2\text{WI}^{-3}].$$

Now $\frac{d}{d\text{I}}\left(\frac{1}{\eta}\right) = 0$ when $\text{WI}^{-2} = a$, i.e. when $\text{W} = a\text{I}^2$,

i.e. when the core loss equals the copper loss; and 2WI^{-3} is positive, as also is $\frac{1}{\text{E}\cos\phi}$; hence $\frac{d^2}{d\text{I}^2}\left(\frac{1}{\eta}\right)$ is positive, i.e. $\frac{1}{\eta}$ is a minimum when the core loss equals the copper loss, i.e. η is a *maximum* under these conditions.

States of Equilibrium of Physical Systems.

It is a law of physics that if the potential energy* of a system is a *maximum* in a state of equilibrium, then that state of equilibrium is an unstable one; while if the potential energy is a *minimum*, the state of equilibrium is stable. For instance, a cone balanced vertically on its apex is in a possible state of equilibrium. But any displacement of the cone from the vertical reduces the potential energy of the cone, and hence the potential energy must be a maximum, and therefore the state of equilibrium is unstable. On the other hand, a cone sitting on its base is in stable equilibrium, because any slight tilting of the cone increases the potential energy of the cone. We can put these results in the rule:

If any small change in the configuration of a system in equilibrium under a conservative system of forces tends to increase the potential energy of the system, the state of equilibrium is a stable one.

If the potential energy (w) of the system can be made to depend on one variable x, then *one* condition of stable equilibrium is that

$$\frac{dw}{dx} = 0.$$

A sufficient second condition is that $\frac{d^2w}{dx^2}$ should be positive.

It was shown in Chapter VII that, by making δx an infinitesimal, $\delta w/\delta x$ differs from dw/dx by an infinitesimal of the first order, hence we can put

$$\frac{\delta w}{\delta x} = \frac{dw}{dx} + \epsilon.$$

$$\text{Now, } \frac{dw}{dx} = 0$$

* i.e. the *mechanical* potential energy.

when the system is in equilibrium, hence

$$\frac{\delta w}{\delta x} = \epsilon,$$

$$\text{i.e. } \delta w = \epsilon \delta x$$

$$\text{or } \delta w = 0,$$

neglecting infinitesimals of higher order than the first, and we get the result:

When a physical system is in equilibrium, the change in its potential energy due to an infinitesimal change in the state of the system is zero, to the first order of infinitesimals.

EXAMPLE *VII*

If p is the amount by which the pressure inside a soap bubble exceeds the atmospheric pressure, r the radius of the bubble, and T the surface tension, show that

$$p = \frac{4\text{T}}{r}.$$

The system, consisting of the soap bubble and the air inside and outside it, is in equilibrium, hence

$$\delta w = 0$$

where w is the potential energy of the system in the equilibrium position.

The increase in the potential energy of the system is the work done against the forces of the system in any infinitesimal change in the configuration of the system.

Let r change to $(r + \delta r)$; then the volume increases by δv and the surface by δS.

The energy of surface tension is TS where T is the surface tension and S the surface, hence the increase in potential energy of surface tension is $2\text{T}\delta\text{S}$ (because there is an inside and an outside surface). The gas in the bubble expands and work is done of amount $p\delta v$, hence the potential energy of the system falls by that amount. The net increase in potential energy is then

$$\delta w = 2T\delta S - p\delta v = 0,$$

$$\text{i.e. } 2T\delta S = p\delta v.$$

$$v = \tfrac{4}{3}\pi r^3.$$

$$\therefore \delta v = 4\pi r^2 \delta r,$$

$$\text{and } S = 4\pi r^2.$$

$$\delta S = 8\pi r \delta r.$$

$$\therefore 2T \times 8\pi r \delta r = p \times 4\pi r^2 \delta r.$$

$$\therefore p = \frac{4T}{r}.$$

A Problem in Engineering Economics.

Certain economic problems can be solved by the methods of this chapter.

EXAMPLE *VIII*

The thermal efficiency of a turbo-alternator is greatly improved by using it with a condenser. On the other hand, the condenser and the auxiliary plant associated with it are expensive in first cost and consume energy in running. The interest on the capital cost and the maintenance charges per annum of the condensing plant, as well as the value of the energy used by it per annum, increase rapidly with the vacuum which the condensing plant gives. A stage is reached when any further increase in the vacuum will bring about a net financial loss. We will take as an example a P-kw. turbine plant, and suppose that the plant draws water for cooling the condenser from a river or the sea.

J. R. Bibbins has published a curve showing how the cost of condensing plants varies with the vacuum.

If A is the capital cost of a 26-in. vacuum plant for a P-kw. turbo-alternator in pounds sterling, the capital cost (C) at a vacuum v in. is given by

$$C = A\left\{1 + \frac{0{\cdot}56(v - 26)}{30 - v}\right\}$$

approximately, between the limits of 26 in. and 30 in. of vacuum.

We will suppose, for the sake of getting a first approximation, that the power taken by the auxiliary plant, both in the boiler- and engine-houses is constant at a per cent, and that the evaporative power of the boilers is constant at q lb. of steam per pound of coal.

The effect of vacuum on the steam consumption of the turbine is given in a set of curves published by W. H. Wallis. These curves show that between 26 in. and 29 in. of vacuum, the percentage reduction in steam consumption (p), is given by

$$p = 4(v - 26) \text{ nearly.}$$

If w is the steam consumption per kilowatt-hour at vacuum v, and w_{26} at 26 in. of vacuum, then

$$w = w_{26}\{1 - 0 \cdot 04(v - 26)\}.$$

It must be clearly understood that this formula is an empirical one which holds between 26 in. and 29 in. of vacuum approximately. Between 29 in. and 30 in. the curve of w against v must clearly bend round and become more or less horizontal because a point will be reached, with increasing v, when the specific volume of the exhaust steam is so great that the exhaust end of the turbine will become choked, and any further increase of vacuum simply goes in overcoming increased exhaust pipe and condenser friction.

We will suppose the coal costs £K per ton, and that the plant is to run at full load only and to be so run for H hours per annum. The interest and maintenance charges in the condensing plant can be put at r per cent per annum.

1. The capital charges amount to

$$\frac{rA}{100}\left\{1 + \frac{0 \cdot 56(v - 26)}{(30 - v)}\right\}£ \text{ per annum.}$$

2. The units of electrical energy (Kelvins) available per annum are

$$P\left(1 - \frac{a}{100}\right)H.$$

3. The steam consumption per kilowatt-hour is

$$w = w_{26}\{1 - 0 \cdot 04(v - 26)\} \text{ lb.},$$

and the cost bill is

$$\left[\frac{Pw_{26}\{1 - 0{\cdot}04(v - 26)\}HK}{2240q}\right]£ \text{ per annum.}$$

4. If we put, for brevity,

$$a = \frac{0{\cdot}56 \times 240rA}{PH\left(1 - \frac{a}{100}\right) \times 100}$$

$$\text{and } \beta = \frac{240w_{26}K}{\left(1 - \frac{a}{100}\right)2240q},$$

we get for the cost, per unit of electricity available, arising from these items (1) and (3), *in pence* (y),

$$y = \frac{a}{0{\cdot}56}\left\{1 + \frac{0{\cdot}56(v - 26)}{(30 - v)}\right\} + \beta\{1 - 0{\cdot}04(v - 26)\}.$$

5. At the most economic vacuum, y must be a minimum, whence

$$\frac{dy}{dv} = 0,$$

$$\text{i.e. } a\left[\frac{1}{30 - v} + \frac{v - 26}{(30 - v)^2}\right] - 0{\cdot}04\beta = 0$$

$$\text{or } \frac{a}{0{\cdot}01\beta} = (30 - v)^2,$$

$$\text{i.e. } (30 - v) = \pm\sqrt{\frac{a}{0{\cdot}01\beta}}.$$

v must be less than 30.

$$\therefore v = 30 - \sqrt{\frac{a}{0{\cdot}01\beta}}.$$

Substituting again for a and β we get

$$\sqrt{\frac{a}{0{\cdot}01\beta}} = 35{\cdot}4\sqrt{\frac{rAq}{w_{26}HKP}}$$

$$\text{and } v = 30 - 35{\cdot}4\sqrt{\frac{rAq}{w_{26}HKP}}.$$

It must be borne in mind that this example takes account of two factors only—the economy arising from a reduction in steam consumption following an improvement in vacuum, and the cost of providing that improvement in condensing plant and its maintenance. There are many other factors which would be taken into account in engineering practice. Some of these are:

1. The reduction in water rate which follows an improvement in vacuum. This brings about a reduction in the size of the boiler plant and of the main steam piping.

2. A smaller circulating water system and smaller circulating water pumps would follow an improvement in vacuum.

3. More power would be taken by the air-pumps or their equivalent at higher vacua.

4. The low-pressure end of the turbine would be larger and more costly at higher vacua.

5. The boiler evaporative power would fall off with higher vacua.

The first two points would lead us to use a higher vacuum, the last three, a lower one.

The problem can only be adequately thrashed out—like most economic problems—with a complete and accurate knowledge of the data and circumstances of each case, and the investigation is best done arithmetically and graphically. These complications notwithstanding, it has become the usual practice to work at a vacuum of about 29 in. in turbine-driven power stations where there is a copious supply of cooling water for the condensers.

Exercise 12

1. Discuss the turning values of

$$\text{(i) } y = x^3 - 6x^2 + 9x - 6.$$
$$\text{(ii) } y = x^5 - 15x^3 + 3.$$

2. What is the radius of the largest cone that can be put inside a sphere made in the form of two accurately-fitting hemispherical cups?

3. How should a rectangular piece of paper be cut so as to fold into a box, without a lid, of maximum volume?

4. Find for what values of θ, $\sin\theta + \cos 2\theta$ has turning values, and ascertain which are maxima and which minima.

5. The corner of a leaf of paper is turned back so as just to reach the other edge of the page. Find when the length of the crease is a minimum.

6. Two ships are sailing uniformly, with velocities u and v, along straight lines converging at an angle θ. Show that if a and b be their initial distances from the point of intersection of the courses, the least distance of the ships apart is

$$\frac{(av - bu)\sin\theta}{(u^2 + v^2 - 2uv\cos\theta)^{\frac{1}{2}}}.$$

7. The path of a projectile is given approximately by

$$y = x\tan\theta - \frac{x^2}{4h\cos^2\theta},$$

where y is the altitude of the projectile, x the horizontal distance of it from the point at which it is fired, and θ is the angle of inclination of the axis of the gun to the horizontal. h is a constant, depending on the initial velocity of the projectile. Find for what value of θ the horizontal range is a maximum. This equation ignores, among other things, air resistance. Would you expect your result to be greater or less than the more correct result, in deriving which air resistance is allowed for?

8. In the example of the soap bubble (p. 229), suppose the bubble carries a charge of electricity e. How would this affect the internal pressure and by how much? [The potential energy of a sphere of radius r carrying a charge e is $\frac{ae^2}{r}$ where a is a constant.]

9. Referring to the condenser problem worked out on p. 230, show that the best vacuum to employ is that at which the rate of increase of the capital charges, per annum, with vacuum equals the rate of decrease of the coal bill, per annum, with vacuum.

Work out completely the case when the capital charges and maintenance of condenser plant are put at 10 per cent per annum; the cost of the condensing plant for a 10,000-kw. plant, £25,000 at 26 in. of vacuum; the evaporative power of the boiler plant, 7·5 lb. of steam per pound of coal; the steam consumption per kilowatt hour at 26 in. of vacuum, 13 lb.; the hours worked 4380 per annum; and the cost of coal £1 per ton.

10. The velocity, v, of the cross head of a reciprocating single-cylinder steam-engine is connected with the angular velocity of the crank by the approximate equation,

$$v = \omega r \left[\sin\omega t + \frac{r}{2l}\sin 2\omega t\right],$$

where ω is the constant angular velocity, r is the crank radius, l the length of the connecting rod, and t is the time measured from the moment when the piston is at the back of the cylinder and just beginning to move outwards. Find for what value of t the velocity of the cross head is a maximum. Calculate this maximum velocity when the speed of the engine is 75 r.p.m., the length of the connecting rod 12 ft., and the crank throw 2 ft. What is the position of the crank at the moment of maximum velocity of the cross head?

CHAPTER XIII

Exponential and Logarithmic Functions

"In order fully to appreciate the brilliancy of Napier's invention and the merit of the work of Briggs and Vlacq, the reader must bear in mind that even the exponential notation and the idea of an exponential function, not to speak of the inverse exponential function, did not form a part of the stock-in-trade of mathematicians till long afterwards. The fundamental idea of the correspondence of two series of numbers, one in arithmetic, the other in geometric progression, which is so easily represented by means of indices, was explained by Napier through the conception of two points moving on separate straight lines, the one with uniform, the other with accelerated velocity. If the reader, with all his acquired modern knowledge of the results to be arrived at, will attempt to obtain for himself in this way a demonstration of the fundamental rules of logarithmic calculation, he will rise from the exercise with an adequate conception of the penetrating genius of the inventor of logarithms."—CHRYSTAL.

In many books of mathematical tables, tables are given of

(*a*) the exponential function, e^x,
(*b*) the logarithmic function, $\log_e x$.

A short table of each function is given at the end of this book.

The *exponential function of* x is the number obtained by raising a certain irrational number $e = 2{\cdot}71828\ldots$, to the power x, for real values of x.

The *logarithmic function of* x, or the natural logarithm of x, is the index of the power to which e must be raised to give the number x.

In elementary algebra, we have

$$\text{if } y = \log_e x,$$
$$\text{then } x = e^y. \quad \text{......................}(1)$$

The theory of these functions is difficult, and proved a formidable stumbling-block in the history of mathematics. The theory is best worked out by methods beyond the scope of this book, and for the present we shall take the tables for granted, and endeavour to ascertain the most important properties of the functions from these tables.

For any pair of values x and y in the logarithmic table there is a pair of values in the exponential (anti-logarithmic) table, such that

$$\text{if } y = \log_e x, \quad \text{....................}(2)$$
$$\text{then } x = e^y, \text{.........................}(3)$$

in fact the two tables are inverse to each other; i.e. in the one, y is regarded as a function of x, which is written $\log_e x$, and in the other, x is regarded as a function of y, written e^y, where y is an index obeying the ordinary laws of indices and e a certain number—the base of the natural logarithm.

Graph of e^y.

A glance at the table of e^y will show that:

(1) when $y = 0$, $e^y = e^0 = 1$;

(2) when y is positive, e^y is positive and greater than 1;

(3) when y $(= -p)$ is negative, $e^{-p} = \frac{1}{e^p}$ = a positive fraction.

Plot a few values of e^y for different values of y, taking values of e^y from the table.

The graph will be found to be as shown in fig. 1.

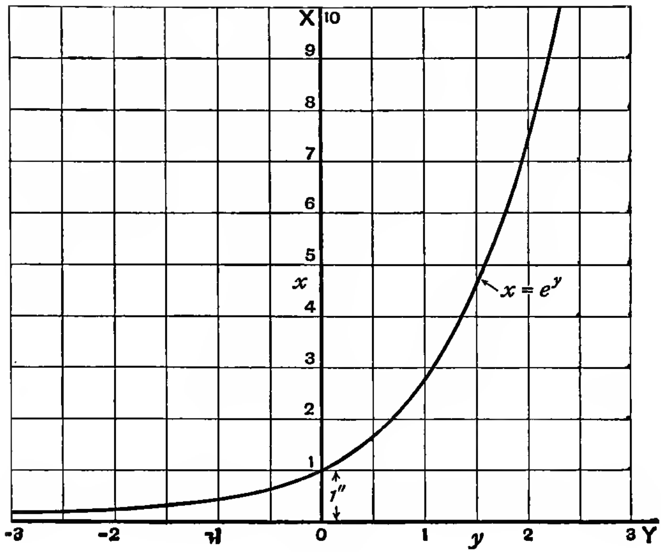


Fig. 1

From the tables or the graph it appears that e^y is never negative.

Graph of $\log_e x$.

The table of $\log_e x$ shows that:

(1) when $x = 1$, $\log_e x = y = 0$;
(2) when $x > 1$, $\log_e x$ is positive;
(3) when $x < 1$, $\log_e x$ is negative.

These facts are also evident from fig. 1.

By plotting a few values of $\log_e x$ for different values of x, the graph will be seen to be as shown in fig. 2, from which it appears that we cannot have the logarithm of a negative number. Draw these two graphs on tracing paper, and show that they can be superimposed on each other. They are therefore the

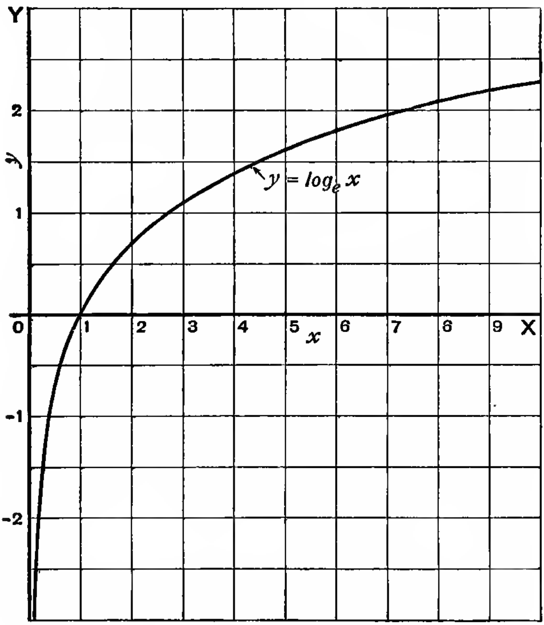


Fig. 2

same curve, but the orientation of the graphs with regard to the axes is different.

These graphs are drawn as smooth regular curves, and, on looking over the numbers given in the tables, there is nothing to suggest that irregularity will be found in any properties possessed by the curves.

One property which a curve has, as a rule, is a definite slope at each point, and it appears that both the graph of e^y and of $\log_e x$ have definite tangent lines and therefore definite slopes at every point on them. We are thus led to consider the *derivatives* of e^y and $\log_e x$.

Derivative of e^y.

$$\begin{aligned} \text{Let } x &= e^y, \\ \text{then } \Delta x &= e^{y+\Delta y} - e^y \\ &= e^y e^{\Delta y} - e^y \\ &= e^y(e^{\Delta y} - 1). \\ \therefore \frac{\Delta x}{\Delta y} &= e^y\left[\frac{(e^{\Delta y} - 1)}{\Delta y}\right] \\ \therefore \underset{\Delta y \to 0}{\text{Lt}}\left(\frac{\Delta x}{\Delta y}\right) &= e^y \times \underset{\Delta y \to 0}{\text{Lt}}\left[\frac{e^{\Delta y} - 1}{\Delta y}\right]. \quad \ldots\ldots(4) \end{aligned}$$

We therefore require the value of

$$\underset{\Delta y \to 0}{\text{Lt}}\left[\frac{e^{\Delta y} - 1}{\Delta y}\right],$$

which the reader will see is simply the value of $\frac{dx}{dy}$ at the point (1, 0), by the definition of the derivative.

This limit is difficult to find rigorously. We can easily see what its value must be from elementary considerations.

1. Plot a few points of the curve $x = e^y$ at intervals of 0·02 from $y = -0{\cdot}1$ to $y = +0{\cdot}1$, and draw the curve through these points. Draw the tangent to the curve at $y = 0$, i.e. at the point (1, 0). It will be found that the tangent line makes an angle of 45° with the axes OX and OY,

$$\text{i.e. } \left(\frac{dx}{dy}\right)_{(1,\,0)} = \tan 45° = 1.$$

But, as we have already remarked, the value of

$$\underset{\Delta y \to 0}{\text{Lt}} \left[\frac{e^{\Delta y} - 1}{\Delta y}\right]$$

is simply the value of dx/dy at the point (1, 0), which we have just seen is 1;

$$\therefore \underset{\Delta y \to 0}{\text{Lt}} \left[\frac{e^{\Delta y} - 1}{\Delta y}\right] = 1. \quad \text{.............(5)}$$

2. Alternatively, we can deduce the value of the limit thus.

A glance at a table of values of e^y shows that, when $y < 1{\cdot}5$,

$$1 + y < e^y < 1 + y + y^2.$$

$$\therefore y < e^y - 1 < y + y^2.$$

$$\therefore 1 < \frac{e^y - 1}{y} < 1 + y.$$

From this inequality it is evident that

$$\underset{y \to 0}{\text{Lt}} \left(\frac{e^y - 1}{y}\right) = \underset{\Delta y \to 0}{\text{Lt}} \left(\frac{e^{\Delta y} - 1}{\Delta y}\right) = 1.$$

$$\therefore \underset{\Delta y \to 0}{\text{Lt}} \left(\frac{\Delta x}{\Delta y}\right) = e^y \times 1.$$

$$\therefore \frac{dx}{dy} = e^y = x.$$

e^y therefore satisfies the equation

$$\frac{dx}{dy} = x, \quad \text{................................(6)}$$

and the derivative of e^y is e^y. This property is highly important, and may be put in words: **the exponential function merely repeats itself by differentiation.** No other function has this remarkable property, which has led to the use of e^y and $\log_e y$ in theoretical calculations.

Derivative of $\log_e x$.

Reverting to equation (4),

$$\frac{\Delta x}{\Delta y} = e^y \times \left[\frac{e^{\Delta y} - 1}{\Delta y}\right].$$

$$\therefore \frac{\Delta y}{\Delta x} = \frac{1}{e^y} \times \left[\frac{\Delta y}{e^{\Delta y} - 1}\right].$$

Now $\Delta y \to 0$ when $\Delta x \to 0$, because $x\ (= e^y)$ is a continuous function of y.

$$\therefore \operatorname*{Lt}_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right) = \frac{1}{e^y} \times \operatorname*{Lt}_{\Delta y \to 0}\left[\frac{\Delta y}{e^{\Delta y} - 1}\right].$$

$$\therefore \frac{dy}{dx} = \frac{1}{e^y} \times 1.$$

$$\therefore \frac{dy}{dx} = \frac{1}{x}, \quad \text{..........................(7)}$$

$$\text{i.e. } \frac{d}{dx}(\log_e x) = \frac{1}{x}. \quad \text{..........................(8)}$$

$$\text{Also } \log_e x = \int \frac{dx}{x} + c. \quad \text{..................(9)}$$

$$\therefore \Big[\log_e x\Big]_a^b = \int_a^b \frac{dx}{x}.$$

$$\therefore \log_e(b) - \log_e(a) = \int_a^b \frac{dx}{x}.$$

Put $a = 1$, then $\log_e(a) = \log_e(1) = 0$.

$$\therefore \log_e b = \int_1^b \frac{dx}{x}. \quad \text{.....................(10)}$$

$\text{Log}_e b$ is a function of b, and therefore $\int_1^b \frac{dx}{x}$ is a function of b. We have therefore succeeded in expressing the logarithmic function as a definite integral.

It is better to keep to x as our variable in $\log_e x$, and putting $b = x$ we get

$$\log_e x = \int_1^x \frac{dx}{x}.$$

The latter form of the integral is open to the objection that the upper limit has the same symbol as the variable.

The upper limit x stands for the *extreme* value of x occurring in the range of integration.

It is more convenient to avoid confusion thus:

$$\int_1^x \frac{dt}{t} \text{ must be the same as } \int_1^x \frac{dx}{x}.$$

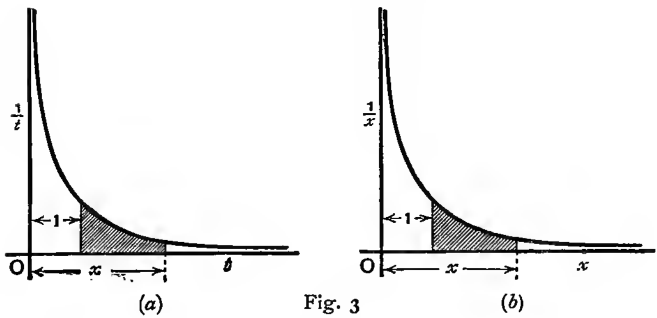


Fig. 3

(Compare figs. 3*a* and 3*b*.)

$$\text{Hence } \log_e x = \int_1^x \frac{dt}{t}. \qquad \text{..............(11)}$$

The two important properties of the logarithmic function have now been found.

1. $\int_1^x \frac{dt}{t} = \log_e x \ (x > 0).$*

2. $\frac{d}{dx}(\log_e x) = \frac{1}{x}.$

* The curve $y = \frac{1}{t}$ or $yt = 1$ is a hyperbola, the asymptotes being the straight lines $t = 0$, $y = 0$. The logarithm of any particular value of t, say x, is the area shown in fig. 3 (*a*). For this reason natural logarithms are often called "hyperbolic logarithms".

To verify graphically that $e = 2{\cdot}718\ldots$

$$\text{We have}\quad y = \log_e x = \int_1^x \frac{dt}{t},$$

$$x = e^y.$$

$$\text{Put}\quad y = 1;\ \text{then}\ x = e.$$

$$\therefore\ 1 = \int_1^e \frac{dt}{t}.$$

To find the value of e approximately, we have to find the position of E (fig. 4), so that the area PAE_1E

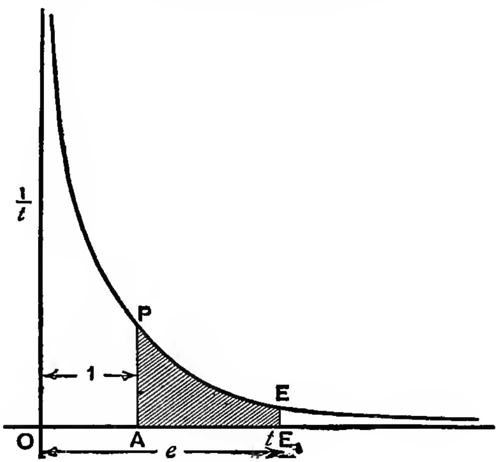


Fig. 4

measures unit area. OE_1 then gives the value of e approximately.

There is no difficulty in getting $e = 2{\cdot}72$ in this way. This figure is very near to the truer value $e = 2{\cdot}718$.

Why e is chosen as the base of the natural logarithms.

The reader may still be at a loss as to why the curious number

$$e = 2{\cdot}71828\ .\ .\ .\ .\ .$$

is used as the base of the logarithms used in theoretical calculations, whereas it would seem so much simpler to stick to the usual base 10. The answer is that 10 can be used if preferred, but its use leads to clumsy multipliers.

Briefly the reasons are these:

1. We found that (p. 240) in finding the derivative of the exponential function, we required the limit

$$\underset{\Delta y \to 0}{\mathrm{Lt}} \left(\frac{e^{\Delta y} - 1}{\Delta y} \right).$$

Suppose we had not chosen base e, but base 10. Everything, in our deduction, would have been the same down to

$$\underset{\Delta y \to 0}{\mathrm{Lt}} \left(\frac{10^{\Delta y} - 1}{\Delta y} \right).$$

To find this limit we should have used the table of ordinary antilogarithms.

On looking up the table, we should find that *we cannot now say that*

$$1 + y < 10^y < 1 + y + y^2,$$

when y is less than a certain value y_0.

This can be said *only* of the exponential table constructed to base e.

2. For many purposes we require the value of

$$\int_1^x \frac{dt}{t},$$

and the value of this is $\log_e x$, a fact which we have ascertained from the table of the exponential function constructed to base e.

Common Logarithms.

There are formulæ corresponding to (5), (6), (8), (11), for $\log_{10} x$ and 10^y, but they are not quite so simple as formulæ (5), &c.

Conversion from natural logarithms to common logarithms and vice versa is quite simple.

The rules are:

1. To convert $\log_e x$ into $\log_{10} x$, multiply by 0·43429, i.e. $\log_{10} x = \log_e x \times 0{\cdot}43429\ldots$

2. To convert $\log_{10} x$ into $\log_e x$, multiply by 2·30259, i.e. $\log_e x = \log_{10} x \times 2{\cdot}30259\ldots$

The proofs of these rules will be found in any book on elementary algebra.

From these figures, it follows that

$$\frac{d}{dx}(\log_{10} x) = \frac{d}{dx}\{\log_e x \times 0{\cdot}43429\ldots\} = \frac{0{\cdot}43429\ldots}{x}, \ldots\ldots(12)$$

by pp. 123, 242.

$$\text{and } \int_1^x \frac{dt}{t} = 2{\cdot}30259\ldots \times \log_{10} x. \ldots\ldots(13)$$

Also $10^x = (e^{\log_e 10})^x = e^{2\cdot30259x}$, and

$$\frac{d}{dx}\left(e^{2\cdot30259\ldots x}\right) = \frac{d}{d(2{\cdot}30259\ldots x)}\left\{e^{2\cdot30259\ldots x}\right\} \times \frac{d(2{\cdot}30259\ldots x)}{dx}$$

$$= e^{2\cdot30259\ldots x} \times 2{\cdot}30259\ldots$$

$$\therefore \frac{d}{dx}(10^x) = 2{\cdot}30259\ldots \times 10^x, \ldots\ldots(14)$$

so that 10^x satisfies the equation

$$\frac{dy}{dx} = 2{\cdot}30259 \times y. \ldots\ldots(15)$$

By taking e as the base, instead of 10, we get rid of these rather clumsy multipliers in our equations.

WORKED EXAMPLES

1. *Integrate* $\int_a^b \frac{c}{(cx+d)}dx$, *where* a, b, c, *and* d *are constants.*

$$\text{Put } z = cx + d. \quad \therefore \frac{dz}{dx} = c,$$

and the given integral becomes

$$\int \frac{dz}{z}.$$

When $x = a$, $z = ca + d$.

$x = b$, $z = cb + d$.

$$\therefore \text{ the given integral } = \int_{ca+d}^{cb+d}\frac{dz}{z}$$

$$= \Big[\log_e z\Big]_{ca+d}^{cb+d}$$

$$= \log_e(cb+d) - \log_e(ca+d)$$

$$= \log_e\left(\frac{cb+d}{ca+d}\right).$$

2. *Differentiate $-\log_e(\cos x)$ with respect to x.*

$$\frac{d}{d\cos x}\{-\log_e(\cos x)\} = -\frac{1}{\cos x}.$$

Put $-\log_e(\cos x) = u$, then

$$\frac{du}{dx} = \frac{du}{d\cos x} \times \frac{d\cos x}{d\,x}.$$

$$\therefore \frac{d}{dx}(-\log_e \cos x) = -\frac{1}{\cos x} \times (-\sin x) = \tan x.$$

NOTE.—From this result, it follows that

$$\int \tan x dx = -\log_e(\cos x).$$

3. *Show that $y = Ae^{mx}$ (where A is an arbitrary constant) satisfies the equation*

$$a_n\frac{d^ny}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \ldots a_1\frac{dy}{dx} + a_0y = 0, \ldots\ldots(16)$$

where a_n, a_{n-1}, a_{n-2}, . . . a_0 are constants, provided m is a root of the equation

$$a_nm^n + a_{n-1}m^{n-1} + \ldots a_1m + a_0 = 0. \ldots\ldots\ldots(17)$$

$$\text{If } y = Ae^{mx},$$

$$\text{then } \frac{dy}{dx} = Ame^{mx},$$

$$\frac{d^2y}{dx^2} = Am^2e^{mx},$$

$$\frac{d^3y}{dx^3} = Am^3e^{mx},$$

$$\frac{d^ny}{dx^n} = Am^ne^{mx}.$$

Substituting in (16) we get

$$(a_n m^n + a_{n-1} m^{n-1} + \ldots + a_0)Ae^{mx} = 0,$$

and any value of m which makes

$$a_n m^n + a_{n-1} m^{n-1} + \ldots a_0 = 0$$

makes Ae^{mx} a solution of the equation, whence the result stated. $A = 0$ also solves the equation, but the solution is useless. In the solution retained, A can be *any* arbitrary constant. This result is highly important, and occurs constantly in science. *Note.*—Equation (16) is a *differential equation with constant coefficients.*

Note on Solutions of Differential Equations.—The following facts regarding *differential equations* may be found useful. They are discussed in full detail, in works devoted to differential equations.

A. If y is a function of x, any equation which expresses a relation between y, the derivatives of y of any order whatever, and x is called a *differential equation.* E.g.

$$\frac{dy}{dx} + \lambda y = 0,$$

where λ is a constant.

B. The *general* solution of a differential equation is the most general relation between x and y, *without any derivatives occurring in it*, which is consistent with the given differential equation.

C. A function of x which satisfies the given differential equation may or may not be the *general* solution. Whether it is the *general* solution or not depends on the number of *arbitrary* constants it contains and the type of the differential equation. $e^{-\lambda x}$ is a solution of the equation in (A), but it is not the *general* solution. It is a *particular* solution.

D. The *order* of a differential equation is the same as the order of the highest derivative it contains when rationalized. Thus, the equation in (A) is of the first order, while (a), p. 249, is of the second.

E. It is proved in works on differential equations that the *general* solution of a differential equation of order n must contain n arbitrary and independent constants and can have no more.

F. In seeking the general solution, therefore, we try to find an equation in x, y, and the requisite number of arbitrary and independent constants, such that when these constants are eliminated we obtain the given differential equation.

Points A to D are definitions of terms. Point E is a theorem which requires proof and has been proved, but the proof is difficult. Point F summarizes the nature of the problem of *solving a differential equation.*

4. *As a particular case of* (16), *solve*

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6 = 0. \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(a)$$

Put $y = Ae^{mx}$ as in example (3), p. 247, and substitute. We get

$$m^2 - 5m + 6 = 0, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(b)$$

$$\text{i.e. } (m-2)(m-3) = 0.$$

$$\therefore m = 2 \text{ or } m = 3,$$

i.e. Ae^{2x} or Be^{3x} satisfies the equation.

By adding these results, we get

$$y = Ae^{2x} + Be^{3x}.$$

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for this value of y and substitute in (a). It will be found that this value of y satisfies the given equation, *whatever values be given to A and B.*

Note there are *two* arbitrary constants in the solution, A and B, which arise from the *two* roots of the quadratic equation for m, namely $m^2 - 5m + 6 = 0$. This condition is characteristic of linear differential equations of the second order.

If $\frac{d^3y}{dx^3}$ were involved, there would be three arbitrary constants; the equation to find m would be a cubic one and the solution would be the sum of three terms, each with an arbitrary constant.

5. *Integrate* $\mathrm{I} = \int\frac{dx}{x^2 - a^2}$, a being a positive number.

$$\mathrm{I} = \int\frac{dx}{x^2 - a^2} = \int y\,dx \text{ where } y = \frac{1}{x^2 - a^2}.$$

The graph of $\frac{1}{x^2 - a^2}$ is shown in fig. 5.

It is traced as follows:

1. Plot the graph of $y = x^2 - a^2$ and take the reciprocals of the ordinates of this graph.

2. When $x^2 = a^2$, the denominator of $1/(x^2 - a^2)$ becomes zero,

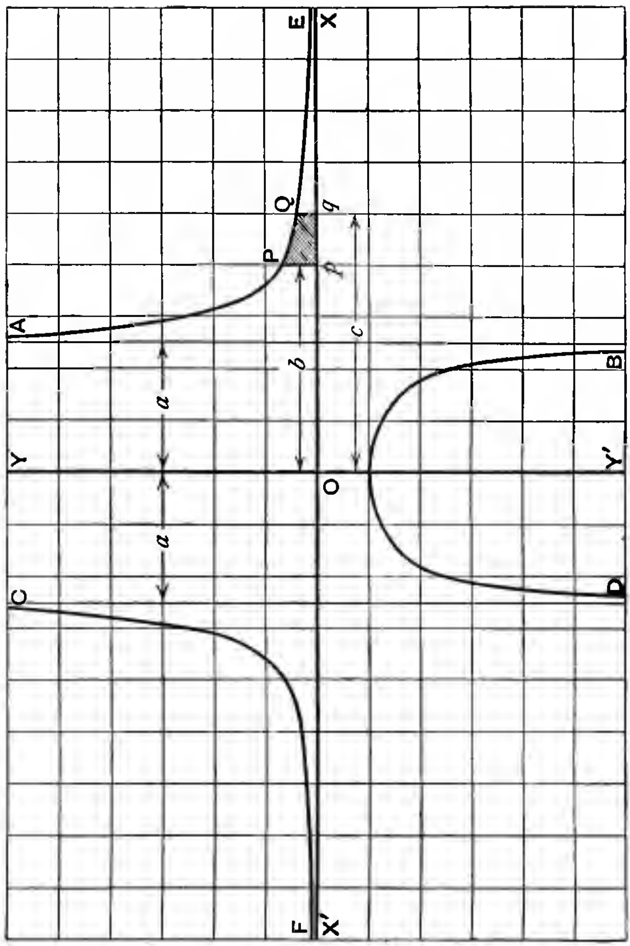


Fig. 5

and the fraction becomes indefinitely great. The straight lines $x = \pm a$, i.e. $x = +a$ and $x = -a$, which are straight lines parallel to OY, and passing through the points $(+a, 0)$ and $(-a, 0)$ respectively, never cut the curve, as there is no point on the curve at a finite distance from OX for which $x = \pm a$. These lines are called *asymptotes*. The line $y = 0$ is another asymptote, since no matter how large x is taken, numerically, $\frac{1}{x^2 - a^2}$ is still more than zero and positive.

3. When $x^2 > a^2$, the denominator is positive; when $x^2 < a^2$, the denominator is negative. The numerator $(+1)$ is always constant and positive. $\therefore$ when x passes through $\pm a$, increasing or decreasing, the value of $1/\overline{x^2 - a^2}$ changes sign, i.e. the graph disappears on one side of the asymptotes $x = \pm a$, and reappears on the other side with the opposite sign.

4. When $x^2 > a^2$, the values of $\frac{1}{x^2 - a^2}$ are positive. $\therefore$ the graph lies on the positive side of OX, when $x > +a$ and $< -a$.

5. When x^2 is very large $\frac{1}{x^2 - a^2} = \frac{1}{x^2}$ nearly, which is a very small positive number. $\therefore$ the graph approaches OX (positive side) when $x \to$ large positive or negative values. In fact (2 above) $y = 0$ is an asymptote.

6. When $x^2 < a^2$, $x^2 - a^2$ is negative and $\frac{1}{x^2 - a^2}$ is negative. $\therefore$ between the asymptotes $x = \pm a$, the graph lies on the negative side of OX, and approaches $-\infty$ when $x \to +a$ or $-a$.

7. When $x^2 < a^2$, $x^2 - a^2 = a^2 - x^2$ *numerically*, and $a^2 - x^2$ is greatest numerically when x^2 is least, i.e. when $x = 0$.

$\therefore \frac{1}{x^2 - a^2}$ is *least* numerically when $x = 0$.

$\therefore \frac{1}{x^2 - a^2}$ must approach nearest to OX for the value $x = 0$,

i.e. $x = 0$ corresponds to a *turning point* in the graph.

There are therefore three branches, AE, BD, and CF.

We must be careful to exclude the values $x = \pm a$ from the range of integration, as when $x = \pm a$, the value of $\frac{1}{x^2 - a^2}$ is indefinitely great as the graph shows, and there is not a definite strip corresponding to a δx which includes $x = \pm a$.

Confine ourselves first to branch AE.

We seek a formula for any area such as PpqQ. $x > a$ everywhere on AE.

$$\frac{1}{x^2 - a^2} = \frac{1}{2a}\left[\frac{1}{x-a} - \frac{1}{x+a}\right].$$

$$\therefore \int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\int\left[\frac{1}{x-a} - \frac{1}{x+a}\right]dx$$

$$= \frac{1}{2a}\int\frac{dx}{x-a} - \frac{1}{2a}\int\frac{dx}{x+a}$$

$$= \frac{1}{2a}[\log_e(x-a) - \log_e(x+a)]$$

$$= \frac{1}{2a}\log_e\left(\frac{x-a}{x+a}\right). \quad \ldots\ldots\ldots\ldots\ldots\ldots(18)$$

Take now the definite integral,

$\int_b^c \frac{dx}{x^2 - a^2}$, where b and c are both positive numbers.

When b and c are both greater than a, the definite integral is

$$\int_b^c \frac{dx}{x^2 - a^2} = \frac{1}{2a}\log_e\left[\frac{(c-a)(b+a)}{(c+a)(b-a)}\right].$$

Now, suppose b and c are not both greater than a. We then pass to the other two branches of the graph.

Branch CF.

If $-c < -b < -a$, in $\int_{-b}^{-c}\frac{dx}{x^2 - a^2}$, dx is negative going from $-b$ to $-c$, but $\frac{1}{x^2 - a^2}$ is positive, x^2 being $> a^2$, within the range of integration; therefore the summation indicated by the integral must be negative, i.e.

$$\int_{-b}^{-c}\frac{dx}{x^2 - a^2} = \left[\frac{1}{2a}\log_e\left(\frac{x-a}{x+a}\right)\right]_{-b}^{-c}$$

$$= \frac{1}{2a}\log_e\left(\frac{-c-a}{-c+a}\right) - \frac{1}{2a}\log_e\left(\frac{-b-a}{-b+a}\right)$$

$$= \frac{1}{2a}\log_e\left(\frac{c+a}{c-a}\right) - \frac{1}{2a}\log_e\left(\frac{b+a}{b-a}\right)$$

$$= -\frac{1}{2a}\left[\log_e\left(\frac{b+a}{b-a}\right) - \log_e\left(\frac{c+a}{c-a}\right)\right]$$

$$= -\frac{1}{2a}\log_e\left[\frac{(c-a)(b+a)}{(c+a)(b-a)}\right].$$

Branch BD.

On this branch $-a < x < +a$, and if we used the formula

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log_e\left(\frac{x-a}{x+a}\right),$$

$(x - a)/(x + a)$ would be a negative number, and we have no logarithm of a negative number.

We can easily avoid this difficulty thus:

$$\int \frac{dx}{x^2 - a^2} = -\int \frac{dx}{a^2 - x^2}, \text{ where } -a < x < +a.$$

$$\frac{1}{a^2 - x^2} = \frac{1}{2a}\left[\frac{1}{a+x} + \frac{1}{a-x}\right].$$

$$\therefore \int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\int\left[\frac{1}{a+x} + \frac{1}{a-x}\right]dx$$

$$= \frac{1}{2a}\int \frac{dx}{a+x} + \frac{1}{2a}\int \frac{dx}{a-x}$$

$$= \frac{1}{2a}\left[\log_e(a+x) - \log_e(a-x)\right]$$

$$= \frac{1}{2a}\log_e\left(\frac{a+x}{a-x}\right).$$

With definite limits we get

$$\int_b^c \frac{dx}{a^2 - x^2} = \frac{1}{2a}\log_e\left[\frac{(a+c)(a-b)}{(a-c)(a+b)}\right],$$

where $-a < b < c < +a$,

$$\text{i.e.} \quad \int_b^c \frac{dx}{x^2 - a^2} = -\frac{1}{2a}\log_e\left[\frac{(a+c)(a-b)}{(a-c)(a+b)}\right]$$

$$= \frac{1}{2a}\log_e\left[\frac{(a-c)(a+b)}{(a+c)(a-b)}\right].$$

The three cases can be combined in one formula if we note that

$$\int \frac{dx}{x} = \tfrac{1}{2}\log_e(x^2),$$

whether x is positive or negative.

$$\text{Therefore} \int \frac{dx}{x^2 - a^2} = \frac{1}{4a}\log_e\frac{(x-a)^2}{(x+a)^2},$$

$$\text{and} \int_b^c \frac{dx}{x^2 - a^2} = \frac{1}{4a}\log_e\left[\frac{(c-a)^2(b+a)^2}{(c+a)^2(b-a)^2}\right].$$

It is important to remember that b and c must be such that neither $+a$ nor $-a$ lies between them. Otherwise the integral is meaningless.

6. *Integration of* $\int \frac{dx}{ax^2+bx+c}$.

The integrand is

$$y = \frac{1}{ax^2+bx+c}.$$

We must examine carefully the denominator of this fraction.

The reader has probably already realized the importance of the vanishing of a denominator. (See Ex. 5, p. 249.)

The quantity ax^2+bx+c equals zero for all values of x that are equal to the roots of the equation $ax^2+bx+c=0$. The equation is quadratic and has two roots. These may be *both* real or *both* complex numbers. If $b^2 > 4ac$ the two roots are real.

Let α and β be these roots. Then, when $x=\alpha$ and $x=\beta$, $ax^2+bx+c=0$. The graph of ax^2+bx+c is a parabola whose axis is parallel to OY.

To show this, transfer the origin from O to O′ where O is the point $\left(-\frac{b}{2a}, 0\right)$, and let O′X′ and O′Y′ be parallel to OX and OY respectively.

If x', y' be the co-ordinates of any point P on the parabola referred to the *new axes*,

$$x' - \frac{b}{2a} = x \text{ and } y' = y.$$

∴ substituting these values in

$$y = ax^2+bx+c, \text{ we get}$$

$$\begin{aligned} y' &= a\left(x'-\frac{b}{2a}\right)^2 + b\left(x'-\frac{b}{2a}\right)+c \\ &= a\left(x'^2-\frac{b}{a}x'+\frac{b^2}{4a^2}\right)+bx'-\frac{b^2}{2a}+c \\ &= ax'^2-bx'+\frac{b^2}{4a}+bx'-\frac{b^2}{2a}+c \\ &= ax'^2-\frac{b^2}{4a}+c \\ &= ax'^2-\left(\frac{b^2-4ac}{4a}\right). \end{aligned}$$

This equation gives the same value for y' whether x' is positive or negative. The graph is therefore symmetrical about the new axis of y, i.e. about the line $x = -\frac{b}{2a} = \frac{\alpha+\beta}{2}$ referred to the original axes (figs. 6, 7, 8).

The position of the vertex is found from the fact that O′V′ is the value of y' when x' is zero, i.e.

$$O'V' = -\left(\frac{b^2 - 4ac}{4a}\right).$$

$(b^2 - 4ac)$ is positive when the roots are real. Also the parabola cuts the OX axis when $y = ax^2 + bx + c = 0$, i.e. at $x = \alpha$ and $x = \beta$, where α and β are the real roots of $ax^2 + bx + c = 0$.

Case I.—Real Roots. $b^2 > 4ac.$

When the roots of $ax^2 + bx + c = 0$ are real, the graph of $ax^2 + bx + c$ is as shown (dotted line) in fig. 6, *when a is positive.* When a is negative, the graph is inverted with respect to OX.

Taking the reciprocal of the ordinate of this graph, for every value of x, we get the heavy line which is the graph of

$$\frac{1}{ax^2 + bx + c}.$$

This graph is of the same nature as that shown in fig. 6, for:

1. When $ax^2 + bx + c = 0$, i.e. when $x = \alpha$ and $x = \beta$, the function

$$\frac{1}{ax^2 + bx + c} \rightarrow \infty.$$

∴ the graph has two asymptotes, $x = \alpha$ and $x = \beta$. $y = 0$ is also an asymptote.

2. When $x < \alpha$ or $> \beta$, the denominator of the fraction is positive (see parabola graph). ∴ the graph lies above the axis X′OX when $x > \beta$ and $< \alpha$.

3. When $\alpha < x < \beta$, the denominator is negative (see parabola graph). ∴ the graph lies below the axis X′OX between these limits of x.

4. Numerically, $\frac{1}{ax^2 + bx + c}$ is least when $ax^2 + bx + c$ is

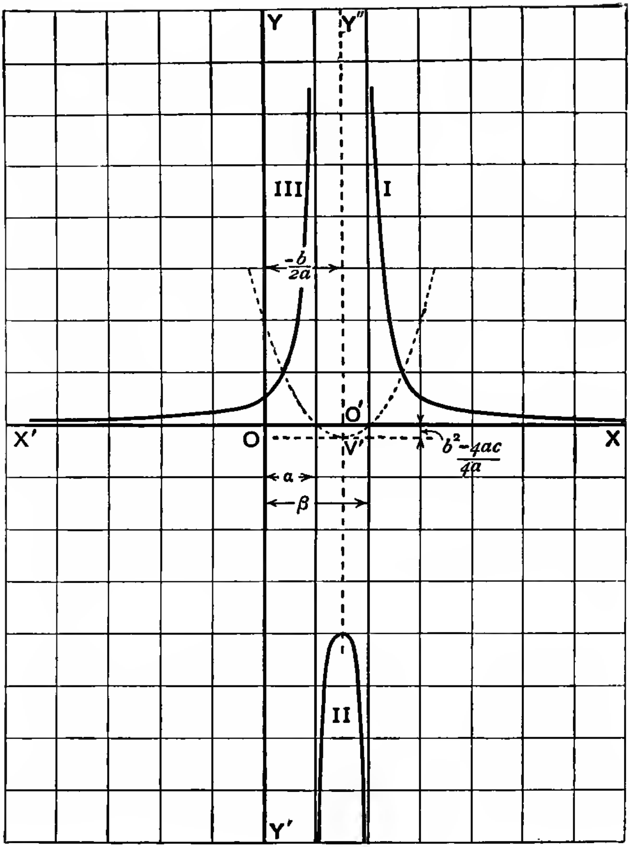


Fig. 6

greatest numerically. This occurs when $x = \frac{\alpha + \beta}{2}$, since the parabola is symmetrical about the line $x = \frac{\alpha + \beta}{2}$.

5. The line $x = \frac{\alpha + \beta}{2}$ is an axis of symmetry of the graph, for it is an axis of symmetry of the parabola.

The graph has therefore three branches which must each be considered separately; the similarity between the graph of this example and the preceding one suggests breaking the expression down to one of the form $\frac{1}{X^2 - A^2}$.

$$ax^2 + bx + c = a\left\{x^2 + \frac{b}{a}x + \frac{c}{a}\right\}$$

$$= a\left\{x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} + \frac{c}{a} - \frac{b^2}{4a^2}\right\}$$

$$= a\left\{\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right\}.$$

Since the roots are real, $b^2 > 4ac$.

$\therefore \frac{b^2 - 4ac}{4a^2}$ is a positive number, which can be written A^2.

$$\text{Put } x + \frac{b}{2a} = X.$$

$$\therefore ax^2 + bx + c = a\{X^2 - A^2\}.$$

$$\therefore \int \frac{dx}{ax^2 + bx + c} = \frac{1}{a}\int \frac{dx}{X^2 - A^2} = \frac{1}{a} \times I,$$

where I is the integral already discussed (5 above).

If $X > A$ or $< -A$,

$$I = \frac{1}{2A} \log_e\left(\frac{X - A}{X + A}\right).$$

$\therefore$ the given integral becomes

$$\frac{1}{\sqrt{b^2 - 4ac}} \log_e\left(\frac{2ax + b - \sqrt{b^2 - 4ac}}{2ax + b + \sqrt{b^2 - 4ac}}\right)$$

on substituting for X and A.

If $-A < X < +A$, the given integral is

$$\frac{1}{\sqrt{b^2 - 4ac}} \log_e\left\{\frac{\sqrt{b^2 - 4ac} - 2ax - b}{\sqrt{b^2 - 4ac} + 2ax + b}\right\} \text{ (p. 253).}$$

Case II.—Imaginary Roots.

$$b^2 < 4ac.$$

In this case the graph of $y = ax^2 + bx + c$ does not cut the OX axis, but its axis still remains parallel to OY (see fig. 7).

If a is positive, y is positive for every value of x, and the graph of $\frac{1}{y}$ is positive. If a is negative, y is negative for every value of x, and $\frac{1}{y}$ is also negative for every value of x.

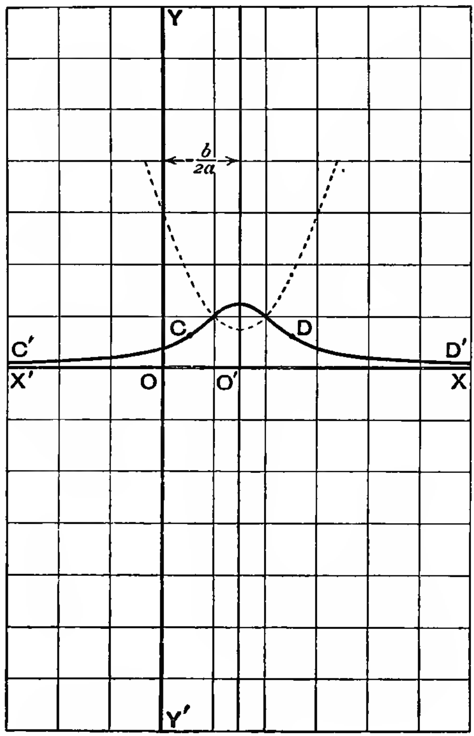


Fig. 7

The general shape is easily seen to be as shown in fig. 7.

$\frac{1}{y}$ is numerically a maximum, when y is numerically a minimum, i.e. at $x = -\frac{b}{2a}$.

The graph is also symmetrical about the line $x = -\frac{b}{2a}$, and $y = 0$ is an asymptote as before. By plotting an actual case as an example, say $y = x^2 - 4x + 13$, it will be found that there are points C and D where the curvature changes from positive to negative. This means that there must be points on $\frac{1}{y}$ for which the curvature is zero. These points are called *points of inflection*. If $z = f(x)$, the curvature is given by

$$\frac{1}{\rho} = \frac{d^2z}{dx^2} \Big/ \left\{1 + \left(\frac{dz}{dx}\right)^2\right\}^{\frac{3}{2}}.$$ See p. 194.

This expression equals zero, when

$$\frac{d^2z}{dx^2} = 0.$$

Points of inflection occur where $\frac{d^2z}{dx^2} = 0$, and $\frac{1}{\rho}$ changes sign.*

Differentiating $z = (ax^2 + bx + c)^{-1}$ twice with respect to x and equating the result to zero, we get

$$3a^2x^2 + 3abx + (b^2 - ac) = 0. \quad \ldots\ldots\ldots\ldots\ldots(19)$$

The roots of this equation are the x co-ordinates of the points of inflection.

The roots of this quadratic are real if

$(3ab)^2 - 4 \times 3a^2(b^2 - ac)$ is positive,

i.e. $9a^2b^2 - 12a^2b^2 + 12a^3c$,

i.e. $12a^3c - 3a^2b^2$,

i.e. $3a^2\{4ac - b^2\}$.

But $4ac > b^2$ is the case we are considering and a^2 is positive, $\therefore 3a^2\{4ac - b^2\}$ is positive, and the roots of (19) are real, i.e.

* A *point of inflection* is a point on a curve where its curvature changes sign. The gradient of the tangent has therefore a turning value at such a point. In the example, this condition is satisfied when $\frac{d^2z}{dx^2} = 0$ and changes sign.

there are two definite points, C and D, at which the curvature is zero.

$ax^2 + bx + c$ can now be put equal to

$$a\left\{\left(x+\frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2}\right\}.$$

∴ the given integral becomes

$$I = \frac{1}{a}\int\frac{dx}{X^2+A^2},$$

where $X = x + \frac{b}{2a}$ and $A = \frac{\sqrt{4ac-b^2}}{2a}$, a positive number.

The integral

$$\int\frac{dx}{X^2+A^2} = \frac{1}{A}\text{ arc tan }\frac{X}{A}\text{ (p. 158).}$$

$$\therefore I = \frac{1}{a}\times\frac{2a}{\sqrt{4ac-b^2}}\text{ arc tan}\left\{\frac{\left(x+\frac{b}{2a}\right)\times 2a}{\sqrt{4ac-b^2}}\right\}$$

$$= \frac{2}{\sqrt{4ac-b^2}}\text{ arc tan}\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right).$$

Case III.—Equal Roots.

In this case $b^2 - 4ac = 0$.

$y = ax^2 + bx + c$ is positive for all real values of x if a is positive, and it is negative if a is negative.

The graph is shown in fig. 8, on the assumption that the parabola lies on the positive side of OX.

The graph is got by reducing the width $(\beta - \alpha)$, fig. 6, to zero, thus making $\alpha = \beta = -\frac{b}{2a}$. The two asymptotes in fig. 6 therefore coincide and appear as one asymptote $x = \alpha$ or β.

The branch II (fig. 6) disappears and we are left with branches I and III. The graph is symmetrical about the axis of the parabola, as before.

(I) $x > \alpha$ and (III) $x < \alpha$.

$$ax^2 + bx + c = a\left\{\left(x+\frac{b}{2a}\right)^2 - \frac{b^2-4ac}{4a^2}\right\}$$

$$= a\left\{\left(x+\frac{b}{2a}\right)^2\right\}.$$

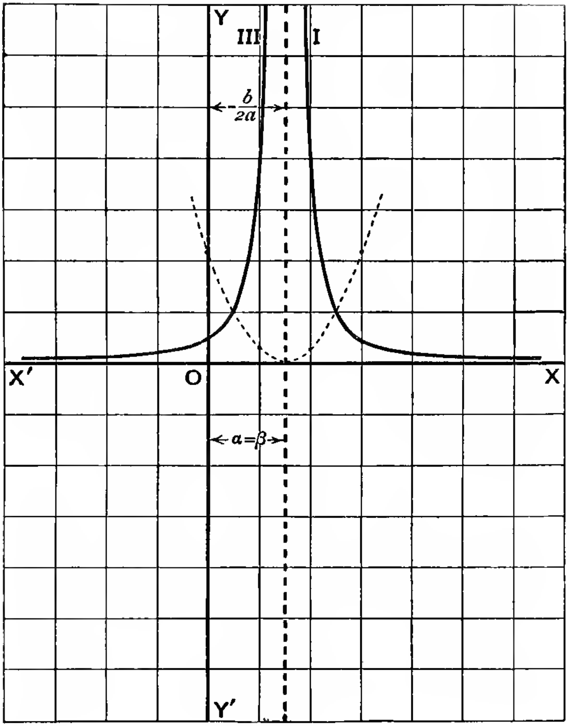


Fig. 8

Put $X = x + \frac{b}{2a}$ as before, then

$$\int \frac{dx}{ax^2 + bx + c} = \frac{1}{a}\int \frac{dX}{X^2}$$

$$= -\frac{1}{aX}$$

$$= -\frac{1}{ax + \frac{b}{2}} = -\left(\frac{2}{2ax + b}\right),$$

a formula which may be used over any range of integration which does not include $x = \alpha \left(= -\frac{b}{2a}\right)$.

Summarizing:

I. When the roots of $ax^2 + bx + c = 0$ are real,

$$b^2 > 4ac$$

and $$\int \frac{dx}{ax^2 + bx + c} = \frac{1}{\sqrt{b^2 - 4ac}} \log_e \left\{ \frac{2ax + b - \sqrt{b^2 - 4ac}}{2ax + b + \sqrt{b^2 - 4ac}} \right\} \quad (a)$$

$$\text{or} = -\frac{1}{\sqrt{b^2 - 4ac}} \log_e \left\{ \frac{\sqrt{b^2 - 4ac} + 2ax + b}{\sqrt{b^2 - 4ac} - 2ax - b} \right\} \ldots\ldots (b)$$

according as $\left(x + \frac{b}{2a}\right)$ does not or does lie between

$$+\frac{\sqrt{b^2 - 4ac}}{2a} \text{ and } -\frac{\sqrt{b^2 - 4ac}}{2a}.$$

II. When the roots of $ax^2 + bx + c = 0$ are imaginary,

$$b^2 < 4ac$$

and $$\int \frac{dx}{ax^2 + bx + c} = \frac{2}{\sqrt{4ac - b^2}} \text{arc tan} \left(\frac{2ax + b}{\sqrt{4ac - b^2}} \right).$$

III. When the roots of $ax^2 + bx + c = 0$ are equal,

$$b^2 = 4ac,$$

$$\int \frac{dx}{ax^2 + bx + c} = -\left(\frac{2}{2ax + b}\right),$$

provided the value of $x = -\frac{b}{2a}$ is not included in any range of integration; i.e.

$$\int_p^q \frac{dx}{ax^2 + bx + c} = -\left[\frac{2}{2ax + b}\right]_p^q,$$

provided p and q are both less than $-\frac{b}{2a}$ or both greater than $-\frac{b}{2a}$.

Integration by Partial Fractions.

Examples 5 and 6 are examples of a method of integration known as *Integration by Partial Fractions.*

If $R(x) = \frac{P(x)}{Q(x)}$, where P and Q stand for polynomials in x, it is often possible to express $R(x)$ as the sum of a series of simpler fractions.

An example will make the method clear.

Suppose

$$R(x) = \frac{3x + 2}{(x - 2)(x - 3)}.$$

Assume that $R(x)$ can be expressed thus—

$$\frac{3x + 2}{(x - 2)(x - 3)} \equiv \frac{A}{x - 2} + \frac{B}{x - 3}$$

where A and B are constants to be found. This relation, if it exists, is an *identity*, i.e. true for *all* values of x.

$$\therefore\ 3x + 2 \equiv A(x - 3) + B(x - 2),$$

$$\text{i.e. } 3x + 2 \equiv (A + B)x - (3A + 2B).$$

These are equal for *all* values of x, if

$$A + B = 3$$

$$\text{and } 3A + 2B = -2,$$

whence $A = -8$ and $B = 11$, so that

$$\int \frac{3x + 2}{(x - 2)(x - 3)}\,dx = \int\left(\frac{11}{x - 3}\right)dx - \int\left(\frac{8}{x - 2}\right)dx$$

$$= 11 \log_e(x - 3) - 8 \log_e(x - 2) + c.$$

Exercise 13

1. Differentiate with respect to x.

(*a*) $\log_e\{x + \sqrt{1 + x^2}\}$.

(*b*) $\log_e[\{x + \sqrt{a^2 + x^2}\}/a]$.

(*c*) $(\log_e x)^n$.

(*d*) $\log_e(\tan x)$.

(*e*) $\log_e\left(\sqrt{\frac{1 - \sin x}{1 + \sin x}}\right)$.

(*f*) $\log_e\left(\sqrt{\frac{1 - \cos x}{1 + \cos x}}\right)$.

(*g*) $e^x \cos x$.

(*h*) $e^{\sin x} \cos x$.

(*i*) $\log_e\sqrt{\sin x} + \log_e\sqrt{\cos x}$.

2. What are the integral formulæ corresponding to (*b*), (*e*), and (*f*) above?

3. Show that

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \log_e\left(\frac{x + \sqrt{x^2 - a^2}}{a}\right),$$

provided $x^2 > a^2$.

4. Show that

$$\int \sqrt{a^2 + x^2}\, dx = \frac{x\sqrt{a^2 + x^2}}{2} + \frac{a^2}{2} \log_e\left(\frac{x + \sqrt{x^2 + a^2}}{a}\right).$$

5. Show that

$$\int \sqrt{x^2 - a^2}\, dx = \frac{x\sqrt{x^2 - a^2}}{2} - \frac{a^2}{2} \log_e\left(\frac{x + \sqrt{x^2 - a^2}}{a}\right),$$

provided $x > a$.

6. Prove that

$$\int_a^b \frac{\log_e x}{x}\, dx = \tfrac{1}{2} \log_e\left(\frac{b}{a}\right) \log_e (ab).$$

7. Integrate $$\int \frac{x}{(x + 2)(x + 3)^2}\, dx.$$

8. Integrate $$\int \frac{x^2}{(x + 2)^2(x + 4)^2}\, dx.$$

9. Integrate $$\int \frac{x^5}{1 + 2x^2}\, dx.$$

10. A pendulum swings under gravity in accordance with the equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0,$$

provided the arc of swing is small, where θ is the inclination of the pendulum to the vertical, g is the acceleration due to gravity, and l is the length of the pendulum.

Find the complete solution of this equation.

If the pendulum is deflected through an angle Θ when

$t = 0$, and is just beginning to move, determine the constants A and B (p. 186) in the complete solution of the equation. Show that it executes harmonic oscillations of period $2\pi\sqrt{\frac{l}{g}}$.

11. A small hollow cubical box floats in a lake. It is ballasted with a layer of lead at the bottom so that one half is immersed. Show that if it is pressed downwards a little, and then left to itself, it will execute vertical harmonic oscillations, to a first approximation.

If M is the mass of the box and a the length of the side, find the period of oscillation. Why would the actual oscillations not necessarily be truly isochronous?

12. In his investigations of the laws of motion of falling bodies, Galileo first assumed that the velocity is proportional to the distance moved through from the start.

Discuss the consequences of this assumption.

13. A copper condenser tube $\frac{3}{4}$ in. outside diameter, and $\frac{1}{16}$ in. thick, is 10 ft. long. Calculate the period of transverse vibration, approximately, assuming the axis of the tube bends into a parabolic form, and that the potential energy of the tube when bent is proportional to the square of the displacement of the axis of the tube from its mean position.

If water is flowing through the tube, what effect does the water have on the period of vibration?

CHAPTER XIV

Some Problems in Electricity and Magnetism

"The little cheesemites held a debate as to who made the cheese. Some thought they had no data to go upon, and some that it had come together by a solidification of vapour, or by the centrifugal attraction of atoms. A few surmised that the platter might have something to do with it, but the wisest of them could not deduce the existence of a cow."—SIR A. CONAN DOYLE.

In this chapter, we shall deal with some applications of the Calculus to electricity and magnetism, but shall first consider a simple mechanical problem which calls for the same kind of mathematical treatment as the electrical problems require.

Consider a fly-wheel of moment of inertia I (lb.-ft.^{2}) spinning about its axis at angular velocity Ω radians per second. Suppose the bearings and windage resistances introduce a retarding torque proportional to the actual angular velocity of the wheel.

It is convenient to measure time from the moment when the motion begins, i.e. when the angular velocity (ω) is Ω, i.e.

$$\text{when } t = 0, \ \omega = \Omega.$$

The angular acceleration of the wheel at time t seconds from the start is

$$\frac{d\omega}{dt}.$$

The bearing friction, &c., introduce a *retarding* torque proportional to ω, say $\kappa\omega$ poundals-feet where κ is a constant.

The *accelerating* torque is theretore

$$-\kappa\omega,$$

hence

$$I\frac{d\omega}{dt} = -\kappa\omega,$$

where I is the moment of inertia about the axis of rotation, by Newton's Laws of Motion applied to a rotating rigid body;

$$\text{i.e. } I\frac{d\omega}{dt} + \kappa\omega = 0,$$

$$\text{or } \frac{d\omega}{dt} + \frac{\kappa}{I}\omega = 0,$$

where κ/I is a (known) constant, say λ.

The equation

$$\frac{d\omega}{dt} + \lambda\omega = 0, \quad \text{.................(1)}$$

therefore determines the angular velocity ω in terms of t, subject to the fact that

$$\omega = \Omega, \text{ when } t = 0.$$

Equation (1) is a *differential equation with constant coefficients*. Its solution has been indicated on p. 248.

The solution is

$$\omega = Ae^{-\lambda t}, \quad \text{.....................(2)}$$

where A is an entirely arbitrary constant. This is the most general function of t that satisfies equation (1).

But we are dealing with a real problem, and a result with an arbitrary constant, to which we can give *any* value we like, is not of much use.

The conditions of the problem require that

$$\omega = \Omega \text{ (a known number), when } t = 0.$$
$$\therefore \Omega = Ae^{-\lambda \times 0} \text{ by substituting in equation (2).}$$
$$\text{i.e. } A = \Omega.$$

To keep our result consistent with the facts, we must therefore give A the *particular value* Ω (A being now no longer arbitrary).

$$\therefore \omega = \Omega e^{-\lambda t}.$$

In this result, Ω and λ are known from the data, hence ω is calculable for any given value of t.

We have therefore found a formula which tells us how the velocity falls as time goes on.

The Electric Circuit.

Two experimental laws govern the flow of currents in closed circuits.

(i) *Ohm's Law.*—When an electric current is flowing along a wire, it experiences a resistance to its flow, which is analogous to ordinary frictional resistance, and electrical pressure is required to drive the current against this resistance.

We can measure currents in amperes and pressure in volts.

It is found *experimentally* that **the pressure required to drive a current along a given wire is proportional to the strength of the current.**

If e is the pressure and i the current, then

$$e \propto i,$$
$$\text{i.e. } e = ri,$$

where r is a constant for the wire in question. r is called *the resistance* of the wire.

It is different for different wires, and depends on the physical state of the wire, e.g. its temperature.

For a given wire, the resistance is the same for all values of i, except in so far as the stronger currents may affect the physical state of the wire, e.g. by heating it and raising its temperature.

The experimental discovery that $e \propto i$ for a given conductor is known as *Ohm's Law.* The product "ri" is often called "the ohmic drop".

(ii) *Faraday's Laws of Induction.*—Before we deal with Faraday's Laws of Induction certain features of the magnetic field must be described.

1. A magnetic N. pole repels another magnetic N. pole with a definite mechanical force which is inversely proportional to the square of the distance between the poles and directly proportional to the product of the *strengths* of the poles.

Let m be the strength of one pole, and
m' ,, ,, the other pole;

then if F is the force of repulsion, and d the distance between the poles,

$$F \propto \frac{mm'}{d^2} = \mu\frac{mm'}{d^2},$$

where μ is a constant.

We will measure F in dynes, d in centimetres, and *define* our pole of *unit* strength thus:

Two poles are of equal and unit strength, if they repel each other with a force of 1 dyne when placed 1 cm. apart *in air.*

In these circumstances,

$$F = 1,\ d = 1,\ m = 1,\ m' = 1.$$

$$\therefore \mu = 1.$$

μ depends on the medium in which the poles are placed.

The *unit* pole strength is *defined* so that μ is *unity in air.*

In what follows, we shall assume that *the medium is air.*

2. *A Magnetic Field* is any region of space in which a magnetic pole experiences magnetic attraction or repulsion.

Consider any point P in such a field (fig. 1). Suppose a N. pole of a magnet of *unit strength* is placed

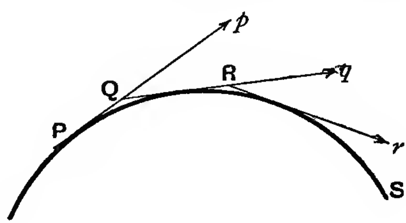


Fig. 1

at P. It will be urged in a certain direction with a definite force. This direction is called *the direction of the field*, and the force, *the strength of the field.*

We can draw an arrow at P (Pp) pointing in the direction in which the unit N. pole is urged. An arrow pointing the opposite way shows the direction in which a S. pole is urged. It is therefore only of interest to have the arrows for one pole. The N. pole is always chosen.

Suppose we move from the point P to a neighbouring point Q on the arrow at P.

At Q, the direction of the field may have altered. Suppose it is given by Qq. Now move to a neighbouring point R on the arrow at Q. R*r* is the direction of the field at R. We can continue in this way to draw lines showing approximately the direction of the field.

Now draw a smooth curve PS, having these arrows as tangents.

When the steps PQ, QR, &c., are made progressively smaller and smaller, the curve PS will tend towards a limiting position and to a limiting form. This limiting curve has the property that its tangent at any point is the direction of the magnetic field at that point. The curve is called *a line of magnetic force.*

We can suppose the whole field to be mapped out by a thread-like system of lines of this kind.

Consider a *uniform* magnetic field, in which the lines of force are all parallel lines. Consider any point P in it and a unit of area (a square centimetre) having P as its centre in a plane at right angles to the direction of the field at P.

The *strength* of the field at P is the force, in dynes, which a unit N. pole at P experiences. Suppose this is B dynes.

We can draw as many lines of force through the unit area at P as we like. Suppose we draw B lines; then B lines of force cut the unit area at P,

i.e. the lines of force per square centimetre of area, perpendicular to the field, at P are the same, numerically, as the strength of the field at P.

The uniform field is therefore represented by a system of parallel lines of force, the number of which is chosen so that B lines cross unit area perpendicular to the lines, at any point in the field.

Varying Field.

When the field varies in strength from point to point, the lines of force will in general be curved, and the number per unit area, perpendicular to the direction of the field, will also vary.

Even in this case, it can be shown that, by properly choosing the number of lines to map out the field, the strength of it, at any point, is given by the above rule.

Oblique Plane Area in Uniform Field.

Suppose we choose any plane area, A sq. cm., the normal to which is inclined at an angle ϵ to the direction of the (uniform) field.

Then (see fig. 2) if PQ is the profile of the given area and QR the profile of its projection in the plane,

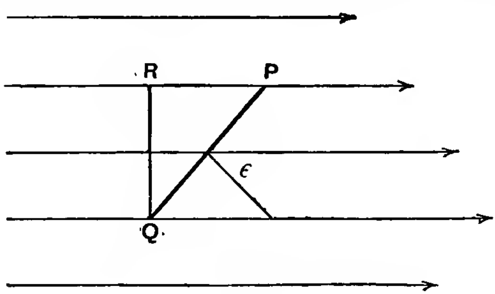


Fig. 2

through Q, perpendicular to the field, the lines crossing the area PQ must be equal to those crossing QR, numerically.

Those crossing QR are

$$B \times (\text{area } QR),$$
$$\text{i.e. } B \times A \cos\epsilon.$$

Hence the lines crossing any plane oblique area A in a uniform field are

$$B_n \times \text{area of A},$$

where B_n is the component of the field strength along the normal to A.

This quantity is called the *magnetic flux* through the area.

The flux through *any* area having the same boundary is evidently the same, as each line cuts both areas.* The important thing is then the *boundary* of the area, and we speak of the flux as "linked" by the curve which forms the boundary of the area.

EXAMPLES

1. Suppose we have a single magnetic pole of strength m. The strength of the magnetic field it sets up at P is given by

$$\frac{m}{r^2} \text{ dynes,}$$

where r is the distance of P from the pole m in centimetres.

Suppose we require the magnetic flux through a spherical surface of radius r.

The direction of the field is everywhere normal to this surface.

$$\therefore \ \phi = 4\pi r^2 \times \frac{m}{r^2} = 4\pi m \text{ lines,}$$

i.e. the surface is cut by $4\pi m$ lines of magnetic force, and 4π lines of magnetic force sprout out from *a unit* positive magnetic pole, as they must all cut a spherical surface enclosing the magnetic pole.

2. Suppose we require the flux through a portion of a spherical surface, radius r, which subtends a solid angle ω at m.

* Provided the two areas compared do not enclose a space containing magnetic matter.

Then, $$\frac{\text{Surface given}}{\text{Whole surface of sphere}} = \frac{\omega}{4\pi}.$$

$$\therefore \phi = \frac{m}{r^2} \times 4\pi r^2 \times \frac{\omega}{4\pi} = m\omega \text{ lines.}$$

Magnetic Flux through Circuit in Variable Field.

Now, let us suppose that the field is a variable one.

Let S be any area, AB (fig. 3), not necessarily plane. Let δs be a small area, at P, inside the

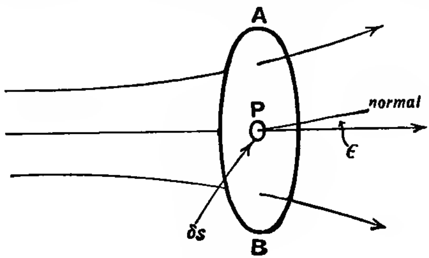


Fig. 3

boundary of S. Suppose this area is small enough to be regarded as a tiny *plane* area.

Let the field at P make an angle ϵ with the normal n to δs at P. Then, the normal component of the strength of the field at P is B $\cos\epsilon$, if B is the strength of the field at P.

The number of lines of force crossing δs is then B $\cos\epsilon\, \delta s$, and the number of lines of force crossing the whole area S is

$$\underset{\delta s \to 0}{\text{Lt}} \left[\Sigma \text{ B} \cos\epsilon\, \delta s\right],$$

the summation being taken over the whole area.

This limit, the existence of which we assume, is denoted by

$$\int \text{B} \cos\epsilon\, ds,$$

or, to show that we are dealing with a summation over an area, by

$$\iint B \cos\epsilon \, ds.$$

An integral of this kind is called a surface integral.*

This quantity—the magnetic flux—is the number of lines of force crossing the whole area, i.e. the number of lines of force tied up, as it were, by the band AB, like stalks of wheat bound into a sheaf. It is denoted by ϕ,

hence $\phi = \int B \cos\epsilon \, ds$

= the surface integral of the normal component of the field strength over the surface.

Magnetic Fields due to Coils carrying Currents.

Oersted discovered, in the year 1820, that coils carrying currents affect magnetic poles in their neighbourhood and therefore set up magnetic fields. The full development of these ideas was largely the work of Ampère and Faraday. It was found experimentally that the magnetic fields are in general very variable from point to point, but that, if the medium in which the field exists is air, the strength of the magnetic field at any point is proportional to the strength of the current in the coil, i.e. if the field strength at a given point is 10 dynes per unit pole

* Here it must be noted s is not a single number which passes continuously from one value to another. We cannot therefore indicate the limits of the integral $\int B \cos\epsilon \, ds$ in the same way as we did with ordinary definite integrals. We therefore make a special statement of the area over which the integral is to be taken.

when the current in the coil is 1 ampere, it is 20 dynes when the current is 2 amperes, and so on.

The flux through *any* coil in the field is therefore proportional to the current in the coil which generates the field. If ϕ_2 is this flux

$$\phi_2 \propto i,$$

where i is the current in the generating coil,

$$\text{i.e. } \phi_2 = \kappa i,$$

where κ is a constant which depends on the shapes, sizes, and relative position of the coils and the number of turns on the generating coil.

It follows as a corollary that the flux ϕ_1 passing through the generating coil itself is proportional to i, the current in the coil, i.e.

$$\phi_1 = \kappa_1 i,$$

where κ_1 is a constant which depends only on the shape, size, and number of turns of wire on the coil.

The important point for us to remember is that, for given coils in given positions, the magnetic flux through any given one of them is proportional to the current in the coil which generates the magnetic field.

Right-handed Screw Law.

Suppose H (fig. 4) is a straight line giving the positive direction of some quantity which has a constant direction, e.g. suppose H is a line of force in a uniform magnetic field. Let P be any plane perpendicular to H in which any closed circuit AB lies.

Suppose AB encircles the line H.

Then the positive direction *in this circuit* is the clockwise direction when we look at the plane of the circuit from the side E in the direction of H.

If P is regarded as a thin piece of board, and if a screw is screwed *into* it at O from the side E, the screw travels in the direction OH. *The screw rotates*

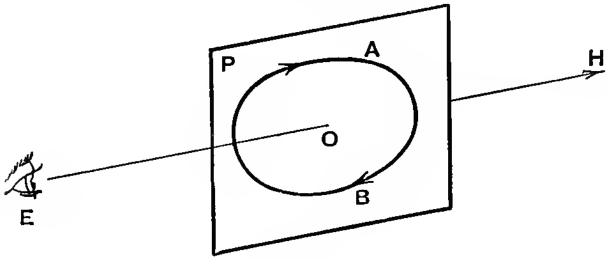


Fig. 4

and travels positively. This is the standard convention known as the *Right-handed Screw Law.*

Faraday's Law.

We can now deal with *Faraday's Law of Electromagnetic Induction.*

Suppose a coil lies in a magnetic field (fig. 5).

Faraday found by experiment that if the field is changing in strength with time, then the coil A has an electro-motive force *induced* in it by the action of the changing magnetic field. This induced e.m.f. is proportional to T—the number of turns in the coil—and to the rate of change of the magnetic flux through the coil. Further, it is in the negative direction in the circuit, as it tends to circulate current in the coil in the negative direction of circulation with regard to the field.

If e is the induced e.m.f., ϕ the magnetic flux passing through the coil A, i.e. "linked" by the coil A, and T the turns on A, then

$$e \propto T\frac{d\phi}{dt}$$

numerically, i.e.

$$e = \kappa T\frac{d\phi}{dt}$$

numerically, where κ is a constant; but as it is in the negative direction of circulation we must give it the negative sign, i.e.

$$e = -\kappa T\frac{d\phi}{dt}.$$

If ϕ is measured as described above, and is in

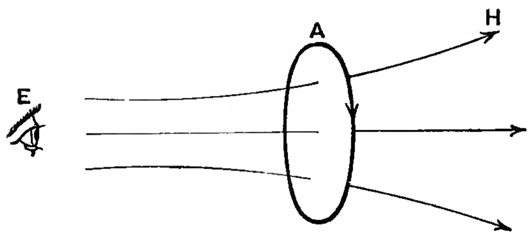


Fig. 5

[dynes per unit pole] × [square centimetres], and if e is measured in volts, then κ has the value 10^{-8}, so that

$$e \text{ (volts)} = -10^{-8} T \frac{d\phi}{dt}, \qquad \text{.........(3)}$$

where $\frac{d\phi}{dt}$ is the rate of *increase* of the *magnetic flux* through the coil.

Equation (3) is the mathematical expression of Faraday's Law of Electro-magnetic Induction, in practical units.

The magnetic field may be due to a magnet or it may be set up by a current in another coil of wire.

For instance, if the coil A carries a clockwise current, when viewed from E, it sets up a magnetic field in the direction shown (the corresponding positive direction) in fig. 6. The e.m.f. induced in B is in the counter-clockwise direction (viewed from the *same* point E).

The two equations

$$e = ri \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

$$\text{and } e = -\mathrm{T}\frac{d\phi}{dt} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(5)^*$$

are of supreme importance in electrical engineering, and underlie the action of all electrical machinery.

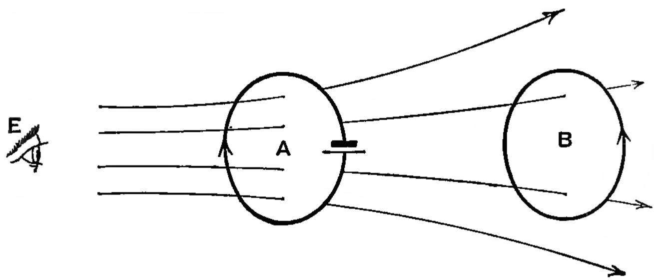


Fig. 6

It should be noticed that they are both the mere symbolical expressions of hard experimental facts, which can be very easily verified with the simplest home-made apparatus.

Self Induction.

But why should the coil B (fig. 6) be affected by changes in the magnetic field produced by A any more than the circuit A itself?

* The reader will bear in mind that in engineer's units (volts, amperes, &c.) this formula must always have 10^{-8} as a factor on the right-hand side.

The answer is that the coil A is also affected. The coil A has an e.m.f. induced in it.

If ϕ_1 is the magnetic flux through A,

$$e = -T_A \frac{d\phi_1}{dt}$$

when T_A is the number of turns in coil A, in series.

This e.m.f. of self induction, as it is called, acts (negative sign) so as to tend to stop the current in A, which is producing the magnetic flux ϕ_1.

The magnetic flux linked with the circuit A, multiplied by T_A, per unit of current in A, is known as the *coefficient of self induction* of A, and is denoted by L. Hence, if i is the current in A, Li is the magnetic flux passing through A and linked with A, multiplied by T_A; for the magnetic field set up by A is everywhere proportional to the current in A (an experimental fact).

$$\therefore T_A \times \phi_1 = Li.$$

$$\therefore T_A \frac{d\phi_1}{dt} = L\frac{di}{dt},$$

since, by experiment again, if the medium is air, L is independent of i. L depends only on the shape, dimensions, and number of turns of wire in the coil A. For a given coil these are constant, and if e is the e.m.f. of self induction, then

$$e = -L\frac{di}{dt}. \qquad \text{.....................(6)}$$

If e is measured in volts, $\frac{di}{dt}$ in amperes per second, then L is measured in *henrys*.

The *henry* is therefore the *flux-turns* linked with a circuit A, when a change of current of 1 ampere per second sets up an induced e.m.f. of 1 volt in that circuit.

The Fall of Current in a Circuit.

Suppose a coil carries a steady currrent C amperes (fig. 7), and that the battery in circuit is suddenly cut out and a resistance cut in instead, we can find the law for the decay of the current with time.

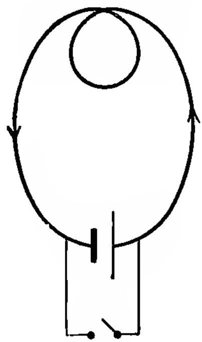


Fig. 7

Suppose we measure time from the instant of cutting out the battery.

Let i amperes be the current after an interval t sec. from this instant. Then the flux through the coil is proportional to i, i.e.

$$\phi = \kappa i \text{ lines},$$

and the e.m.f. of self induction in volts is

$$e = -10^{-8}\,T\frac{d\phi}{dt},$$

where T is the number of turns on the coil,

$$= -10^{-8}\,T\kappa\frac{di}{dt}$$

$$= -L\frac{di}{dt},$$

where L is the self inductance of the coil in henrys. The minus sign simply means that this e.m.f. *opposes* the flow of the current which generates the magnetic field, i.e. it is a "back" e.m.f.

Suppose E is the applied e.m.f. in the circuit in the direction in which the current is flowing. The applied e.m.f., minus the "back e.m.f. of induction", as it is called, is numerically equal to the net e.m.f. in the circuit in the direction of the current. This net e.m.f. is, in its turn, equal numerically to the "ohmic drop" by Ohm's Law. Hence we get the equation,

$$E - L\frac{di}{dt} = ri,$$

where r is the resistance of the whole circuit.

But when the battery is cut out, E = 0.

$$\therefore\ L\frac{di}{dt} + ri = 0,$$

$$\frac{di}{dt} + \frac{r}{L}i = 0, \quad \text{.................(7)}$$

and $i = C$ when $t = 0$. ....(8)

We have, then, to solve

$$\frac{di}{dt} + \frac{r}{L}i = 0, \quad \text{....................(9)}$$

subject to i being equal to C when $t = 0$.

This is exactly the same mathemathical problem as arose in the problem of the rotating fly-wheel, for r

and L are constants, and the quotient of them may therefore be replaced by a single constant, say λ. The only difference is that we have current i instead of angular velocity ω and our new λ is a different constant. Compare equations (1) and (2).

The complete solution is therefore

$$i = Ce^{-\lambda t}.$$

Compare equation (2).

The similarity of the mathematical result is interesting.

It suggests that (1) L is a kind of an electric inertia corresponding to the mechanical moment of inertia I; (2) r is an electrical resistance analogous to mechanical resistance; and (3) the electric current is analogous to angular velocity.

We know by mechanics that the *kinetic energy* of a fly-wheel of moment of inertia I is

$$\tfrac{1}{2}I\omega^2,$$

where ω is the angular velocity.

By analogy, the kinetic energy of an electric current should be

$$\tfrac{1}{2}Li^2.$$

This analogy is found to be tenable. We can work out all the electrical phenomena of circuits by *the principle of the conservation of energy* if we suppose that:

1. The k.e. of the circuit is $\frac{1}{2}Li^2$.
2. (Resistance × current) is equivalent to a retarding torque which deprives the system of its kinetic energy.
3. Current is equivalent to angular velocity.

The "retarding torque" takes energy out of the system at a rate

$$(ri \times i) \text{ watts.}$$

(Compare $\kappa\omega \times \omega$ = torque × angular velocity, in mechanics.)

Gain of energy per second $= -ri^2$, as it is a loss and so negative.

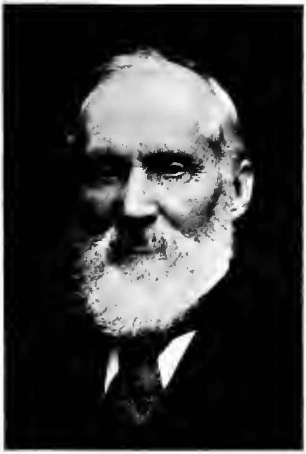
$$\therefore \frac{d}{dt}[\tfrac{1}{2}Li^2] = -ri^2.$$

$$\therefore Li\frac{di}{dt} = -ri^2.$$

$$\therefore L\frac{di}{dt} + ri = 0.$$

We thus arrive at the same equation by *mechanical principles*.

These analogies have been examined very closely by physicists, and have been found to be more than mere analogies. The whole theory of ordinary mechanics and electricity can be brought under one treatment of "generalized mechanics". These mechanical analogies, being true ones, are most useful in forming clear ideas of the behaviour of electric currents. For instance, it should be difficult to stop suddenly a current in a highly inductive circuit, just as it is dangerous to try to stop suddenly a heavy fly-wheel rotating at high speed. Every electrical engineering student knows that a bad arc may take place if the field coils of a direct-current dynamo are suddenly opened, and that the usual field switch is designed to throw a "kicking" coil, i.e. a resistance, into circuit with the field coils when the source of excitation is switched off.



KELVIN (1824–1907)
From a photograph by Annan

CLERK MAXWELL (1831–1879)
From the portrait by Lowes Dickinson
(Trinity College, Cambridge)

FARADAY (1791–1867)
From the portrait by T. Phillips, R.A.
(National Portrait Gallery)

JOULE (1818–1889)
From the portrait by John Collier
(Royal Society)

PHYSICISTS

The reason for the arc is obvious from Faraday's Equation.

The induced e.m.f. is proportional to $\frac{d\phi}{dt}$, i.e. to $\frac{di}{dt}$, and if the current is suddenly broken, $\frac{di}{dt}$, i.e. the *rate* at which the current changes, is very large, and so the induced e.m.f. is very heavy. Several thousand volts can easily be induced by the sudden breaking of a highly inductive field circuit, and these pressures may break down the insulation of the coils and put them out of action until they are rewound—hence the use of the kicking coil to avoid breaking the circuit *suddenly*, and to dissipate the kinetic energy of the field current gradually—not explosively as an arc does.

Rise of Current in a Circuit.

Consider the circuit sketched in fig. 8.

C is a cell; L, a coil of self inductance L henrys; R, a resistance of r ohms (supposed large compared

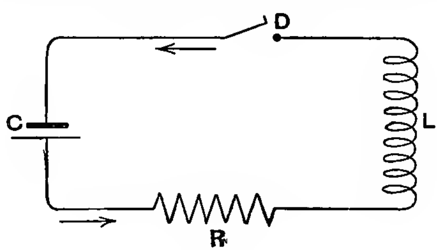


Fig. 8

with the internal resistance of the battery and the resistance of the rest of the circuit, so that we may suppose the total resistance concentrated in R). Let

E be the constant pressure given by the cell. Suppose the switch D is closed. The current begins to rise.

Let i be its value at time t ($t = 0$ at instant of closing D).

Then the induced pressure opposing E is $L\frac{di}{dt}$.

$$\therefore E - L\frac{di}{dt} = \begin{cases}\text{pressure available for driving} \\ \text{current against resistance.}\end{cases}$$

$$\therefore E - L\frac{di}{dt} = ri.$$

$$\therefore L\frac{di}{dt} + ri = E,$$

$$\text{i.e. } \frac{di}{dt} + \frac{r}{L}i = \frac{E}{L}. \quad \ldots\ldots\ldots\ldots\ldots\ldots(10)$$

One solution of this equation is

$$i = \frac{E}{r};$$

$$\text{for } \frac{di}{dt} = 0 \text{ if } i = \frac{E}{r}, \text{ a constant,}$$

$$\text{and } \frac{E}{r} \times \frac{r}{L} = \frac{E}{L}.$$

Suppose E were zero, then, by p. 267, we know that $i = Ae^{-\frac{r}{L}t}$ solves the equation,

$$L\frac{di}{dt} + ri = 0.$$

Add the two solutions;

$$i = Ae^{-\frac{r}{L}t} + \frac{E}{r}.$$

On differentiating this expression for i with respect to t, and substituting in (10), we see that this equation is a solution of the differential equation (10).

$$\text{But } i = 0 \text{ when } t = 0.$$

$$\therefore 0 = A + \frac{E}{r}. \quad \therefore A = -\frac{E}{r}.$$

$$\therefore i = \frac{E}{r}\left(1 - e^{-\frac{r}{L}t}\right),$$

and every constant in this equation is known.

The graph of i in terms of t is as shown in fig. 9.

As t increases, $e^{-\frac{r}{L}t} = \frac{1}{e^{\frac{r}{L}t}}$ clearly decreases, and the higher the resistance the quicker the current attains, practically, its maximum value, while the

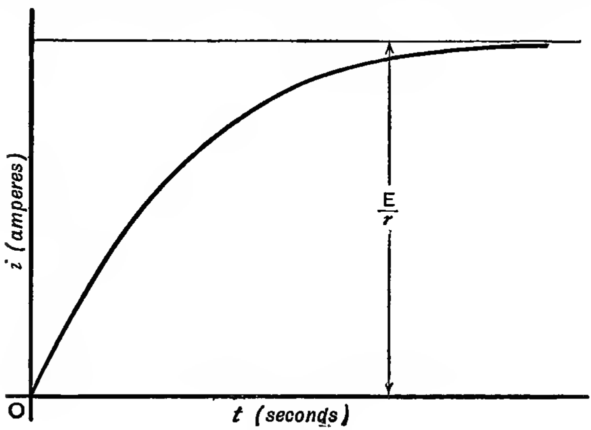


Fig. 9

higher the self inductance, the slower is the rise in current. If the coil is wound round an iron core the flux produced by a given current is very much greater than it is in air. Highly inductive circuits are therefore to be expected from coils wound on iron cores.

EXAMPLE

Suppose a ring is made from a round bar of soft iron, 2 in. diameter, so that the internal diameter of the ring is 12 in. Suppose there are 500 turns of No. 18 B.W.G. copper wire wound closely on it. Let this coil be connected through a key to the terminals of a secondary cell of negligible internal resistance and giving about 1·5 volts. The resistance of such a coil is about 1·3 ohms, and the maximum current

$$\frac{1 \cdot 5}{1 \cdot 3} = 1 \cdot 15 \text{ amperes.}$$

This current is not sufficient to cause "saturation" of the iron, so that for all currents between 0 and 1·15 amperes, the magnetic induction will be nearly proportional to the current producing it for steady currents. If the iron is of low hysteresis, we may suppose that changes in magnetic induction follow closely changes in current and are proportional thereto. We can thus give an equivalent self inductance to the coil. In the case stated, it would be about 1 henry.

Suppose the circuit is completed.

$$i = \frac{E}{r}\left(1 - e^{-\frac{r}{L}t}\right), \text{ at time } t,$$

$$\frac{r}{L} = \frac{1 \cdot 3}{1} = 1 \cdot 3,$$

$$\frac{E}{r} = \frac{1 \cdot 5}{1 \cdot 3} = 1 \cdot 15,$$

$$i = 1 \cdot 15\,(1 - e^{-1 \cdot 3t}) \text{ amperes.}$$

Plot this curve and show that the current attains 95 per cent of its maximum value in about 2·3 sec. This example has been chosen to illustrate the order of magnitude of the different quantities involved in a particular case where the current would rise slowly. Coils with iron cores, such as that described, are used as "choking coils" in alternating-current engineering. Another example of coils in which currents change slowly when "made" or "broken" is the field coils of dynamos. Slow rise or fall of current is always associated with highly inductive circuits.

In actual fact, the current would not rise exactly in accordance with the formula obtained above, because:

(*a*) Though the iron is unsaturated, the magnetic flux is not quite proportional to the current producing it even under steady conditions. If the iron is saturated when the full current is attained, the flux is not even approximately proportional to the current.

(*b*) The question is further complicated by

(i) Hysteresis, which causes the change in flux to lag behind the change in current producing it.

(ii) Eddy circuits, which are induced in the body of the iron and have a similar effect.

The formula is practically correct for coils in air, but r/L always has a much higher value than unity for such coils, and so the current rises or falls very quickly indeed—under $\frac{1}{20}$ sec., say.

These time effects go by the general name of *transient phenomena*.

Damping of Instruments.

Consider a small body, such as the small magnet carrying the reflecting mirror of a galvanometer. Suppose, further, that the body is suspended by a thread, so that when it is displaced from its position of equilibrium the thread brings into play a restoring torque proportional to the angular displacement. Then the rate of angular deceleration is proportional to the displacement. Accordingly, if θ is the displacement,

$$-\frac{d^2\theta}{dt^2} = \omega^2\theta \text{ } (\omega^2 \text{ a positive constant}).$$

$$\therefore \frac{d^2\theta}{dt^2} + \omega^2\theta = 0.$$

A solution of this equation we know to be

$$\theta = A \sin\omega t, \text{ where A is any constant.}$$

This is the most general solution which permits θ to be zero when t is zero. The motion is a simple harmonic motion. Now, suppose there is, in addition to the restoring torque, a drag on the needle, which for small velocities is proportional to the angular velocity of the needle (compare pp. 266, 267). This drag will contribute an additional retarding torque proportional to the angular *velocity*, say equal to $2\kappa\frac{d\theta}{dt}$. The equation for the angular deceleration of the needle therefore becomes

$$-\frac{d^2\theta}{dt^2} = 2\kappa\frac{d\theta}{dt} + \omega^2\theta,$$

$$\text{i.e. } \frac{d^2\theta}{dt^2} + 2\kappa\frac{d\theta}{dt} + \omega^2\theta = 0 \quad \text{..........(11)}$$

Suppose that κ is small, so that $\omega > \kappa$.

Try as a solution of this equation,*

$$\theta = Ae^{-\kappa t}\sin\sqrt{\omega^2 - \kappa^2}\,t. \quad \text{...........(12)}$$

It will be found, on differentiating this expression to find $\frac{d\theta}{dt}$ and $\frac{d^2\theta}{dt^2}$ and on substituting in (11), that it does satisfy the equation.

This is the equation of *damped* harmonic motion.

κ is called the "attenuation" constant.

The graph is of the type shown in fig. 10.

It is evident from (12) that the oscillations are *isochronous* and of period

$$2\pi/\sqrt{\omega^2 - \kappa^2}, \text{ i.e. } Oc'' = c''c''' = \&\text{c., where}$$

$$Oc'' = \frac{2\pi}{\sqrt{\omega^2 - \kappa^2}} = T'.$$

* A tentative solution, such as this, is suggested by more advanced considerations. It is not obvious and is not meant to be.

Damping makes the *time* of swings a little longer and gradually suppresses them. These conclusions are obviously reasonable.

At the points B, B′, B″, &c., where the growth of

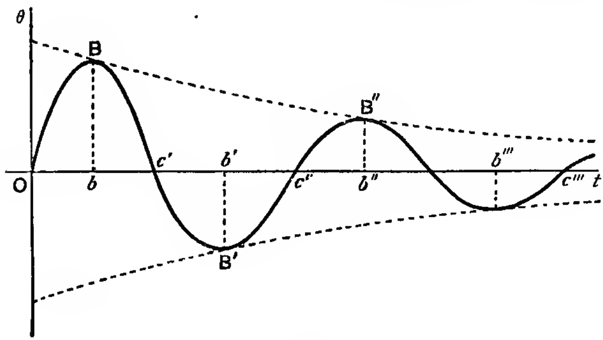


Fig. 10

θ reverses its sense, the tangent to the graph is parallel to Ot.

$$\therefore \frac{d\theta}{dt} = 0 \text{ at B, B}', \text{ B}'', \&\text{c.}$$

Another way of looking at this point is that θ in

$$\theta = \mathrm{A}e^{-\kappa t} \sin\sqrt{\omega^2 - \kappa^2}\, t \quad\ldots\ldots\ldots\ldots(13)$$

is the displacement of the needle from the zero line, and $d\theta/dt$ is therefore the measure of its velocity.

When the needle is not moving any farther outwards, its velocity outwards must be zero.

$$\therefore \frac{d\theta}{dt} = 0 \text{ at B, B}', \text{ B}'', \&\text{c.}$$

From (13) $d\theta/dt = 0$ gives

$$\mathrm{A}e^{-\kappa t}\{\sqrt{\omega^2 - \kappa^2} \cos\sqrt{\omega^2 - \kappa^2}\, t - \kappa \sin\sqrt{\omega^2 - \kappa^2}\, t\} = 0.$$

Hence, if t' is the value of t which makes this equation hold,

$$\sqrt{\omega^2 - \kappa^2} \cos\sqrt{\omega^2 - \kappa^2}\, t' = \kappa \sin\sqrt{\omega^2 - \kappa^2}\, t'.$$

$$\therefore \tan\sqrt{\omega^2 - \kappa^2}\, t' = \frac{\sqrt{\omega^2 - \kappa^2}}{\kappa}.$$

$$\therefore t' = \frac{1}{\sqrt{\omega^2 - \kappa^2}}\left[m\pi + \text{arc tan}\frac{\sqrt{\omega^2 - \kappa^2}}{\kappa}\right], \quad (14)$$

where m is any positive or negative integer or zero.

This formula gives the values of Ob, Ob', Ob'', &c. (fig. 10).

Let t have the value t_1 = Ob,
or t_2 = Ob'.

By (14) $t_2 = t_1 + \frac{\pi}{\sqrt{\omega^2 - \kappa^2}}$, on putting $m = 0$ and 1 respectively.

$$\text{Hence, } bB = Ae^{-\kappa t_1} \sin\sqrt{\omega^2 - \kappa^2}\, t_1,$$

$$\text{and } b'B' = Ae^{-\kappa t_2} \sin\sqrt{\omega^2 - \kappa^2}\, t_2$$

$$= Ae^{-\kappa\left(t_1 + \frac{\pi}{\sqrt{\omega^2 - \kappa^2}}\right)} \sin\left[\sqrt{\omega^2 - \kappa^2}\left\{t_1 + \frac{\pi}{\sqrt{\omega^2 - \kappa^2}}\right\}\right]$$

$$= Ae^{-\kappa t_1} e^{-\frac{\kappa\pi}{\sqrt{\omega^2 - \kappa^2}}} \sin\{\sqrt{\omega^2 - \kappa^2}\, t_1 + \pi\}$$

$$= -Ae^{-\kappa t_1} e^{-\frac{\kappa\pi}{\sqrt{\omega^2 - \kappa^2}}} \sin\sqrt{\omega^2 - \kappa^2}\, t_1.$$

$$\therefore \frac{b'B'}{bB} \text{ (numerically)}$$

$$= e^{-\frac{\kappa\pi}{\sqrt{\omega^2 - \kappa^2}}} = e^{-\lambda}, \ \lambda \text{ positive. } \quad \ldots\ldots\ldots\ldots\ldots(15)$$

$$\therefore \log\left(\frac{bB}{b'B'}\right) = \frac{\kappa\pi}{\sqrt{\omega^2 - \kappa^2}} = \lambda \text{ (positive).}$$

λ is evidently the same for any other pair of half amplitudes, e.g. $b'B'/b''B''$.

From equation (13) it is clear that the "damped" period is

$$\frac{2\pi}{\sqrt{\omega^2 - \kappa^2}},$$

hence the *undamped* period would be

$$T = \frac{2\pi}{\omega} \text{ (put } \kappa = 0 \text{ in (13)).}$$

$$\therefore \frac{T}{T'} = \frac{\sqrt{\omega^2 - \kappa^2}}{\omega} = \frac{\kappa\pi}{\lambda\omega} \text{ by (15).}$$

$$\therefore T = \frac{\pi}{\sqrt{\pi^2 + \lambda^2}} \times T'. \qquad \text{.................(16)}$$

The object of deliberately "damping" an instrument needle is to enable the instrument to be used again after a reading, without having to wait a long time for the needle to come to rest.

Since the *damped* motion is definitely related to the corresponding undamped motion by the equations already derived, we can easily calculate the undamped motion of the needle from observation of its damped motion.

Ballistic Galvanometer.

In the *Ballistic Galvanometer*, for instance, the quantity of electricity held by a condenser can be measured by discharging the electricity through the instrument and observing the ensuing swing of the needle on each side of the zero (neutral) position.

In this instrument

$$Q = \mu T a,$$

where Q is the quantity of electricity discharged in coulombs,

μ, a constant for the instrument,

T, the *undamped* period of oscillation of the needle,

and a, the maximum *undamped* swing of the needle following the discharge.

Let a' be the first observed deflection (damped) on one side of the zero,

and β' be the first observed deflection (damped) on the other side of the zero,

and let T' = period of oscillation of the needle (damped).

a', β', and T' are all observable quantities.

To connect these with T and a, we may proceed thus:

$$T = \sqrt{\frac{\omega^2 - \kappa^2}{\omega^2}}\,T'.$$

The motion of the needle is given by

$$\theta = Ae^{-\kappa t}\sin\sqrt{\omega^2 - \kappa^2}\,t.$$

Let t_1 be the time when the first maximum swing takes place.

Then
$$a' = Ae^{-\kappa t_1}\sin\sqrt{\omega^2 - \kappa^2}\,t_1$$
$$= Ae^{-\kappa t_1}\sin\left\{\text{arc tan}\frac{\sqrt{\omega^2 - \kappa^2}}{\kappa}\right\}$$

putting $m = 0$ in (14).

$$\therefore a' = Ae^{-\kappa t_1}\frac{\sqrt{\omega^2 - \kappa^2}}{\omega} = \frac{\Omega}{\omega}e^{-\kappa t_1},$$

where Ω is the initial value of $d\theta/dt$ (last line of p. 291), which is the same whether there is damping or not. To find a, we put $\kappa = 0$.

$$\therefore a = \frac{\Omega}{\omega}, \text{ or } a = a'e^{\kappa t_1}.$$

$$\therefore\ Q = \mu T a$$
$$= \mu T a' e^{\kappa t_1}.$$

We now have to get rid of $e^{\kappa t_1}$.

$$\text{We have } \frac{a'}{\beta'} = e^{\lambda},$$

$$\text{where } \lambda = \frac{\kappa\pi}{\sqrt{\omega^2 - \kappa^2}} \text{ by (15).}$$

$$\text{Also } t_1 = \frac{1}{\sqrt{\omega^2 - \kappa^2}} \text{ arc tan} \frac{\sqrt{\omega^2 - \kappa^2}}{\kappa}$$
$$= \frac{\lambda}{\kappa\pi} \text{ arc tan} \frac{\pi}{\lambda}.$$
$$\therefore\ Q = \mu T\, a' e^{\frac{\lambda}{\pi} \text{ arc tan} \frac{\pi}{\lambda}}.$$

If $\frac{\lambda}{\pi}$ is very small, we have, since $\frac{\pi}{\lambda}$ is very large,

$\text{arc tan}\frac{\pi}{\lambda} = \frac{\pi}{2}$ nearly.

$$\therefore\ Q = \mu T a' e^{\frac{\lambda}{2}}.$$
$$= \mu T' a' \sqrt{\frac{a'}{\beta'}} \text{ nearly,} \quad \ldots\ldots..(17)$$

since $\sqrt{\frac{a'}{\beta'}} = e^{\frac{\lambda}{2}}$, and T is very nearly equal to T′.

This result gives Q in terms of the measurable quantities

$$T',\ a',\ \beta',$$

and μ, the constant of the instrument.

The only approximation made in this calculation is that λ is small compared to π, i.e. the damping is slight.

Exercise 14

1. A fly-wheel of moment of inertia I is spinning about a shaft at angular velocity Ω.

It is then brought to rest by a retarding couple proportional to its angular velocity.

Show that its speed falls in accordance with the equation

$$I\frac{d\omega}{dt} + \mu\omega = 0,$$

where ω is the angular velocity at time t and μ is a constant.

Solve this equation, and find the value of the arbitrary constant in the solution, in terms of Ω.

If the fly-wheel is 6 ft. in diameter and has a mass of 1 ton, find the value of μ necessary to bring the fly-wheel to half-speed in 5 min., if $\Omega = 150$ r.p.m.

2. Show that, if an electric circuit lies in a magnetic field so that it embraces a magnetic flux ϕ, the change in the magnetic flux taking place in an interval of time $(t_2 - t_1)$ is proportional to the amount of electricity displaced round the electric circuit, and is given in absolute electromagnetic units by

$$\phi_2 - \phi_1 = \frac{r}{T}Q,$$

where ϕ_1 is the magnetic flux at time t_1, ϕ_2 at t_2, Q the quantity of electricity which flows past any section of the circuit in the interval $(t_2 - t_1)$, r the resistance, and T the number of turns of the circuit.

3. Using the result of question 2, show that, if a ballistic galvanometer is used to measure the amount of electricity displaced in the circuit, then the change of magnetic flux $= \mu a$ where a is the first observed deflection on one side of the zero, and μ is a constant for the *circuit*, including the galvanometer.

4. A straight solenoid lies with its length in the direction of a uniform magnetic field in air of 100 lines per

square centimetre. The number of turns on the solenoid is 500 and its diameter is 7·5 cm.

The solenoid is connected to a ballistic galvanometer and is turned suddenly through a right angle.

The first fling observed on one side is 65 divisions.

Find the change of flux per scale division. If the resistance of the circuit were halved, what would be the observed fling, assuming no damping effect.

5. A coil of wire consisting of 100 turns of fine wire, the diameter of the coil being 4 cm. and its resistance 50 ohms, is situated in a magnetic field of 100 lines per square centimetre. This field is reduced to zero at a uniform rate, the total time taken to reduce it being a tenth of a second. Calculate the magnitude of the current induced in the small coil in amperes.

6. A coil of 20 turns of wire in the form of a circle 30 cm. in diameter, rotates 2000 times per minute about a vertical diameter. Find the maximum e.m.f. in volts induced in the coil. (Horizontal component of the earth's magnetic field = ·18 lines per square centimetre.)

7. In the ballistic galvanometer, when there is no damping, the equation of the motion of the needle (compare pp. 289, 290) is

$$\frac{d^2\theta}{dt^2} + \omega^2\theta = hi,$$

where the notation is the same as in equation (11) above, h being a constant and i the current.

Integrate this equation, subject to the conditions that i is constant and that θ and $\frac{d\theta}{dt}$ are 0 when t is 0. Prove that i is proportional to the extreme deflection, and that the needle always keeps to the same side of its undisturbed position.

CHAPTER XV

Some Problems in Chemical Dynamics

"The idea of an atom has been so constantly associated with incredible assumptions of infinite strength, absolute rigidity, mystical actions at a distance, and indivisibility, that chemists and many other reasonable naturalists of modern times, losing all patience with it, have dismissed it to the realms of metaphysics, and made it smaller than 'anything we can conceive'. But if atoms are inconceivably small, why are not all chemical actions infinitely swift?"—LORD KELVIN (1870).

The extract at the head of this chapter, which is taken from an article Lord Kelvin wrote on Atoms (*Nature*, March, 1870), gives an idea of the state of atomic theory in 1870. The position to-day is:

Number of molecules in 1 c. c. of any gas at 0° C. and 760 mm.	$= 2 \cdot 7 \times 10^{19}$.
Number of molecules in 1 gm. molecule of a gas	$= 6 \cdot 1 \times 10^{23}$.
Average distance apart of adjacent molecules at 760 mm.	$= 3 \times 10^{-6}$ cm.
Mass of the hydrogen molecule ...	$= 3 \cdot 3 \times 10^{-24}$ gm.
Average velocity of hydrogen molecules at 0° C.	= 1690 m. per sec.
Radius of hydrogen molecule (considered as a sphere)	$= 1 \cdot 2 \times 10^{-8}$ cm.

These numbers* are taken, with slight modifications, from Jean's *Dynamical Theory of Gases* (1916), and show the great strides that have been made in fifty years in atomic theory.

It is interesting to compare these figures with Lord Kelvin's estimate in 1870. He found that

(*a*) The radius of a molecule of a gas could not be less than 10^{-9} cm.

(*b*) The number of molecules per normal centimetre could not be greater than 6×10^{21}.

Subsequent research has evidently confirmed both of these predictions.

The first measurement of the velocity of a chemical reaction was made about 1850 by Wilhelmy, who measured the rate of inversion of cane sugar by acid. The first step towards a satisfactory explanation of what governs the velocity of chemical reactions was taken by Guldberg and Waage when they established their celebrated law—*The Law of Mass Action*.

Many compound gases are found to decompose into their elements if they are left standing sufficiently long. The decomposition is measurable within a reasonable time if the gases are kept within suitable temperature limits. Among the many instances which have been carefully studied is arsine, AsH_3 (Van 't Hoff, *Studies in Chemical Dynamics*).

* Such numbers are often curiously printed—for instance,

$2{\cdot}75 \times 10^{19}$ appears as **2·75 × 10^{19}**.

The important figure is the index, and the memory is helped by thinking in a logarithmic notation. Thus—

Log. number of molecules per normal c.c. ...	...	= 19
Log. radius of molecule	...	= − 8,
and so on.		
It is natural then to find log. mass of molecule	...	= − 24.

Suppose a vessel A contains arsine (fig. 1). The vessel is connected to a pressure-gauge CF, which can

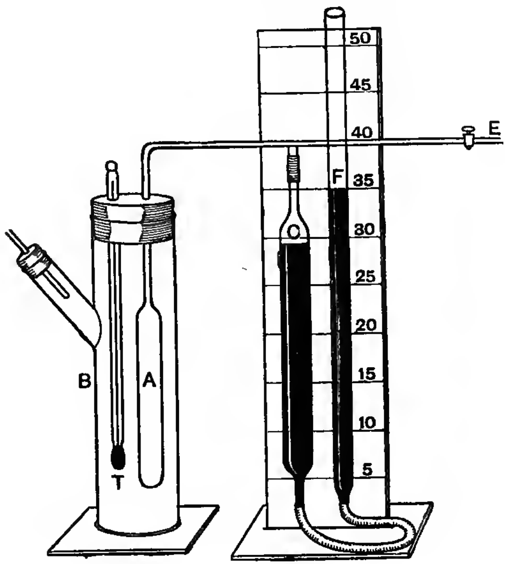


Fig. 1

be raised or lowered bodily so as to maintain the gas at constant volume. The pressure-gauge reads any change of pressure which may occur. The vessel A is heated to a fairly high *constant* temperature, say in the vapour of boiling diphenylamine (310° C.). Initially A contains only arsine. As time goes on, the pressure is found to rise (at constant volume and temperature), indicating that something must have happened to the molecules in A. If the number of molecules in A had not changed, there could be no rise in pressure at constant volume and temperature.

Further, free hydrogen is found to make its appearance and there is no evidence of *atomic* hydrogen. A mirror of arsenic is deposited on the walls of the vessel.

The simplest molecular equation which would account for these experimental facts is

$$2AsH_3 = 2As + 3H_2.$$

Arsenic cannot exist as a vapour at 310° C. and approximately atmospheric pressure, and therefore it condenses out and is deposited as a mirror on the walls of the vessel. The equation indicates that for every 2 molecules of arsine which disappear we get 3 molecules of hydrogen.

Let n be the number of molecules of a gas X per litre,
m, the molecular weight of the gas,
w, the mass of the hydrogen atom, in grammes;
then nmw is the mass of the gas X per litre in grammes.
$\therefore nw$ = the mass of the gas X per litre in gramme-molecules, i.e. when the molecular weight in grammes is taken as the *unit* mass instead of the gramme.

The gramme-molecule is often called the "mol". Hence the mass of X per litre in "mols" *divided by the mass of the hydrogen atom* (a constant) is the number of molecules of X per litre.

It is usual to measure all masses in chemical dynamics in mols, i.e. we use a different unit of mass for different substances, the unit mass used being *the molecular weight in grammes*. The masses so measured are proportional to the number of molecules present per litre, and the constant of proportionality (w) is *the same for all substances*. The mass of X present in mols per litre is called *the concentration of* X. This number is proportional to the number of molecules in unit volume. At a given temperature and pressure, all gases have the same concentration, by Avogadro's Theory.

Let a be the original concentration of AsH_3 at time $t = 0$,
and x, the amount (in mols per litre) of AsH_3 transformed in time t.

Then $(a - x)$ is the concentration at time t,

$$\text{and } \frac{\frac{3}{2}x + a - x}{a} = \frac{p}{p_0}, \quad \text{.................(1)}$$

where p is the pressure at time t,
and p_0 ,, ,, $t = 0$ (i.e. at start),

since, by the kinetic theory of gases, the pressure at constant temperature and volume is proportional to the number of molecules present.

$$\therefore\ x = 2a\left(\frac{p - p_0}{p_0}\right). \quad \text{...............(2)}$$

That is to say, the rise of pressure is directly proportional to the amount of AsH_3 decomposed.

The change in pressure therefore gives a direct measure of the progress of this reaction.

Dissociation might arise in two ways:

(*a*) By collision of pairs of AsH_3 molecules.
(*b*) By a kind of molecular explosion.

In the former case, at least two molecules must take part in each collision; while in the latter, a single molecule is alone concerned in each explosion.

In the former case, the dissociation could be perhaps expressed by

$$2AsH_3 \rightarrow 2As + 3H_2, \quad \text{.............(3)}$$

while, in the latter, we might have

$$AsH_3 \rightarrow As + 3H, \quad \text{...............(4)}$$

the free hydrogen atoms uniting in accordance with the equation,

$$6H \rightarrow 3H_2. \quad \text{....................(5)}$$

As there is no experimental evidence of the presence of *atomic* hydrogen, it must be supposed to become molecular hydrogen with great rapidity if it is formed at all, and the reaction as a whole must be dominated by

$$AsH_3 \rightarrow As + 3H,$$

the reaction $6H \rightarrow 3H_2$ going on much more quickly.

The essential difference between equation (3) and equation (4) is that in the former *two* molecules of AsH_3 take part in each discrete transformation, whereas in equation (4) one molecule of AsH_3 only is involved. The resulting products are the same in both cases, namely, deposited arsenic and hydrogen, mixed with undecomposed arsine. The two cases can be distinguished by considering the influence the number of reacting molecules has on the speed of the reaction.

Number of Reacting Molecules.

Suppose there are n_a molecules of a gas A and n_b of a gas B per litre. The number of collisions between pairs of molecules per second must depend on n_a and n_b.

Suppose n_a is doubled, without altering the mean velocity of the A and B molecules (i.e. suppose the partial pressure of the A molecules is doubled). There will then be twice as many molecules for each of the B molecules to run into, and if the ratio which the number of actual collisions leading to dissociation bears to the number of possible collisions does not change, there will be twice as many collisions with accompanying dissociations per second.

Now, suppose n_b is also doubled. There will now be four times as many collisions per second, i.e. the number of collisions with dissociation per second is proportional to $n_a \times n_b$.

There is no reason why A and B should be essentially different molecules. The point is that the number of bimolecular collisions with dissociation per second is proportional to $n_a \times n_b$.

If $n_a \equiv n_b$, the number of bimolecular collisions with dissociation per second is proportional to n_a^2.

In the same way, the trimolecular collisions with dissociation per second would be proportional to $n_a \times n_b \times n_c$, and the m-molecular collisions to $n_a \times n_b \times \ldots$ to m factors.

Now, the number of collisions with dissociation per second gives at once the amount of A (or B or C) which appears (or disappears) per second.

Let x be the concentration of A at time t.

Then $nw = x$, where n is the number of molecules of A per litre, and w the mass of the hydrogen atom, in grammes.

$$\therefore\ n = \frac{x}{w},\ w \text{ being constant.}$$

Now, $\frac{dn}{dt}$ is the rate at which molecules of A make their appearance, hence $\frac{dn}{dt}$ new molecules of A appear in the gas *per second*, and if each one arises from a collision, then $\frac{dn}{dt}$ is the number of collisions with dissociation per second.

$$\text{Since } n = \frac{x}{w},$$

$$\frac{dn}{dt} = \frac{1}{w}\frac{dx}{dt},\ w \text{ being constant.}$$

But $\frac{dx}{dt}$ is the rate at which the concentration of A increases per second, i.e. $\frac{dx}{dt}$ measures the speed at which the reaction goes forward, and it is proportional to the number of collisions with dissociation per second. As the number of collisions with dissociation per second must be proportional to

$$n_a \times n_b \times \ldots \text{ to } m \text{ factors,}$$

for a reaction depending on a *m*-molecular collision, we have

$$\frac{1}{w}\frac{dx}{dt} \propto n_a \times n_b \times \ldots \text{ to } m \text{ factors,}$$

$$\text{i.e. } \frac{dx}{dt} = \kappa(n_a \times n_b \times \ldots \text{ to } m \text{ factors}), \ldots(6)$$

where κ is a constant which is called the *reaction constant.*

In the case of arsine, then, if the dissociation is bimolecular, and if x is the concentration of arsine at time t, we get

$$-\frac{dx}{dt} = \kappa x^2. \ldots\ldots\ldots\ldots(7)$$

(The negative sign arises because arsine is disappearing.)

If the reaction goes forward as a series of monomolecular explosions, the number of explosions per second will depend on the number of molecules per litre, and it is reasonable to assume that by doubling the number of molecules present, we should double the number of explosions, i.e. if x denotes the number of mols of A present per litre at time t, then

$-\frac{dx}{dt}$, the speed of the reaction, is proportional to n_a;

$$\text{i.e. } -\frac{dx}{dt} \text{ is proportional to } x;$$

$$\text{i.e. } -\frac{dx}{dt} = \lambda x \text{ where } \lambda \text{ is a constant} \ldots\ldots(8)$$

The equations (7) and (8) are the mathematical expressions for the two cases, and are governed by the one rule:

Velocity of reaction is proportional to

$$n_a \times n_b \times n_c \times \ldots \qquad (9)$$

This relationship is known as the *Law of Mass Action*. It was enunciated by Guldberg and Waage. In words, it may be put:

> **The velocity of reaction is proportional to the product of the concentrations of the reacting molecules.**

The law can be deduced directly from the principles of thermodynamics.

Equation (9) is the simplest way in which the law can be stated, and is the key to many formulæ of chemical dynamics.

Collecting our results, we see that if AsH_3 decomposes, by collisions, in accordance with

$$2AsH_3 \rightarrow 2As + 3H_2,$$
$$\text{i.e. } AsH_3 + AsH_3 \rightarrow 2As + 3H_2,$$

the law of mass action gives the velocity of the reaction as

$$\frac{dx}{dt} = -\kappa x^2, \qquad (10)$$

where x is the concentration of arsine at time t, and κ is a constant (for the particular temperature of the experiment).

On the other hand, if the reaction is governed by

$$AsH_3 \rightarrow As + 3H,$$

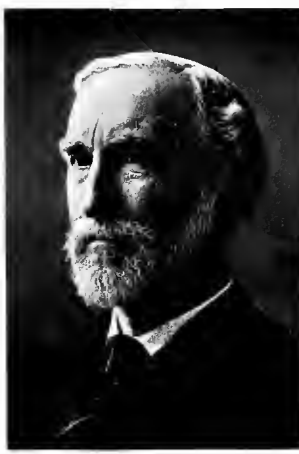
we should have

$$\frac{dx}{dt} = -\lambda x, \qquad (11)$$

GRAHAM (1805-1869)
From a lithograph by W. Bosley

VAN'T HOFF (1852-1911)
From a photograph

GULDBERG (1836-1902)
From a photograph



GIBBS (1839-1903)
From a photograph

PHYSICAL CHEMISTS

where λ is a constant (for the particular temperature of the experiment).

Equations (10) and (11) are different, hence the Law of Mass Action gives us a means of testing whether the reaction is monomolecular or bimolecular.

If equation (10) holds, we get

$$\kappa \frac{dt}{dx} = -\frac{1}{x^2}.$$

$$\therefore \kappa t = -\int \frac{1}{x^2} dx + c,$$

$$\text{i.e. } \kappa t = \frac{1}{x} + c.$$

To find c, we know that when $t = 0$, $x = a$, the initial concentration.

$$\text{Hence } 0 = \frac{1}{a} + c.$$

$$\therefore c = -\frac{1}{a},$$

$$\text{and } \kappa t = \frac{1}{x} - \frac{1}{a},$$

$$\text{i.e. } t = \frac{1}{\kappa}\left(\frac{1}{x} - \frac{1}{a}\right). \quad \text{................}(12)$$

If equation (11) holds, we get

$$-\frac{dx}{dt} = \lambda x,$$

$$\text{i.e. } -\lambda \frac{dt}{dx} = \frac{1}{x}.$$

$$\therefore -\lambda t = \log_e x + c.$$

To find c, we know that when $t = 0$, $x = a$.

$$\therefore 0 = \log_e a + c.$$

$$\therefore c = -\log_e a.$$

$$\therefore \lambda t = \log_e\left(\frac{a}{x}\right),$$

$$\text{i.e. } t = \frac{1}{\lambda}\log_e\left(\frac{a}{x}\right). \quad \ldots\ldots\ldots\ldots\ldots\ldots(13)$$

We thus get the two equations,

$$\kappa = \frac{1}{t}\left(\frac{1}{x} - \frac{1}{a}\right) \text{ if (10) holds,}$$

$$\text{i.e. } \kappa = \frac{1}{at}\left(\frac{a}{x} - 1\right), \quad \ldots\ldots\ldots\ldots\ldots\ldots(14)$$

$$\text{and } \lambda = \frac{1}{t}\log_e\left(\frac{a}{x}\right) \text{ if (11) holds.}$$

To test these equations, we require data of x and t. These data have been obtained experimentally (see textbooks of Physical Chemistry or Van 't Hoff's book already referred to).

An experiment gave

Time in Hours.		Pressure (mm.)*
0	....	784·84
3	....	878·50
4	....	904·05
5	....	928·02
6	....	949·28
7	....	969·08
8	....	987·19

EXAMPLE *I*

From these data calculate:

(1) The ratio a/x by equation (2),

(2) $\frac{1}{t}\left(\frac{a}{x} - 1\right)$,

(3) $\frac{1}{t}\log_e\left(\frac{a}{x}\right)$,

* Lehfeldt, *A Textbook of Physical Chemistry.*

and see whether (2) or (3) is more nearly constant over the range of the experiment.

What conclusion do you draw from your result?

EXAMPLE *II*

As another example of a gas reaction, we will consider the decomposition of phosphine.

The molecular reaction would be

$$4PH_3 \rightarrow P_4 + 6H_2,$$

if phosphorus is present as vapour. Molecules of phosphorus are tetra-atomic at temperatures considerably above the boiling-point of the element. They become diatomic at very high temperatures. If, however, the temperature is not very high, the reaction might go forward as

$$PH_3 \rightarrow P + 3H; \qquad (15)$$

$$\left.\begin{array}{l}\text{then } 4P \rightarrow P_4 \\ \text{and } 2H \rightarrow H_2\end{array}\right\}, \qquad (16)$$

the latter two reactions being assumed to be very rapid compared to the first.

Test which set of equations is the more likely from the following data:—

Time in Hours.		Pressure of Gaseous Mixture.*
0	....	715·21 mm.
7·83	....	730·13 ,,
24·17	....	759·45 ,,
41·25	....	786·61 ,,
63·17	....	819·96 ,,
89·67	....	855·50 ,,

Assume that when 4 molecules of PH_3 disappear we get 6 molecules of hydrogen. Then equation (2) gives x/a in terms of p. Use equation (13) on the assumption of a monomolecular reaction, and to get the formula for a quadrimolecular reaction, use

$$\frac{dx}{dt} = -\mu x^4, \text{ where } \mu \text{ is a constant}, \qquad (17)$$

and integrate it.

* König. (Quoted from Lewis, *A System of Physical Chemistry*.)

A Reversible Reaction.

As an example of another, and more complicated, type of reaction, take the decomposition of hydriodic acid.

When hydriodic acid is left standing it dissociates into hydrogen and iodine, and the dissociation can be measured if the gas is kept at a suitable temperature, so that the dissociation is neither too fast nor too slow. If the gas initially consists solely of hydriodic acid, the reaction proceeds until a certain percentage of the acid is decomposed. It then ceases.

Similarly, if we start with hydrogen and iodine vapour in molecular proportions, these gases combine until a certain definite percentage of hydriodic acid is present. This percentage is the same as that attained by dissociation, and depends only on the temperature of the reaction.

The molecular equation is

$$2HI = H_2 + I_2. \qquad \text{(18)}$$

If a is the initial concentration of the hydriodic acid and x the *amount transformed* in time t, $(a - x)$ is the concentration of hydriodic acid at time t.

Hence, if equation (18) represents the reaction, we have

$$2HI \rightarrow H_2 + I_2$$

as the equation of dissociation, and

$$\frac{dx}{dt} = \kappa(a - x)^2 \qquad \text{(19)}$$

by the law of mass action.

But at time t we have molecules of hydrogen and iodine present, and there appears to be no reason why these molecules should not recombine in accordance with the law of mass action. The experimental fact that the reaction is reversible supports this idea.

When x mols of HI have dissociated we have

$$\frac{x}{2} \text{ mols of } H_2 \text{ and } \frac{x}{2} \text{ mols of } I_2 \text{ present,}$$

and by recombination they unite to *decrease* x, so that the time rate of *increase* of x is not $\kappa(a - x)$,[2] but

$$\frac{dx}{dt} = \kappa(a - x)^2 - \frac{\lambda x^2}{4}. \quad \ldots\ldots\ldots\ldots(20)$$

Equilibrium would be reached when

$$\frac{dx}{dt} = 0,$$

i.e. if $x = x_\epsilon$ for equilibrium,

$$\kappa(a - x_\epsilon)^2 = \frac{\lambda x_\epsilon^2}{4},$$

$$\text{i.e. } \frac{4(a - x_\epsilon)^2}{x_\epsilon^2} = \mu \text{ where } \mu = \frac{\lambda}{\kappa} = \text{a constant,}$$

$$\text{or } \frac{a - x_\epsilon}{x_\epsilon} = \frac{\sqrt{\mu}}{2};$$

$$\text{i.e. } \frac{x_\epsilon}{a} = \text{a constant.} \quad \ldots\ldots\ldots\ldots\ldots\ldots(21)$$

At a given temperature, then, a constant fraction of the original HI should be transformed, and this fraction is independent of the pressure of the gas used in the experiment (i.e. of a). It is easy to show that the final or equilibrium composition will be the same, whether we start with pure hydriodic acid or with a mixture of hydrogen and iodine vapour in molecular proportion.

Bodenstein found at 448° C.:

Original pressure, HI	$\frac{1}{2}$	1	$1\frac{1}{2}$	2 atmos.
x_ϵ/a	20·19 %	21·43 %	22·25 %	23·06 %

These figures show a 15 per cent increase in x_ϵ/a for a 300 per cent increase in pressure.

EXAMPLE

Taking $x_\epsilon/a = 0{\cdot}2200$, calculate $\mu = \left(\frac{\lambda}{\kappa}\right)$ at 448° C.*

We now have to consider what happens *before the equilibrium condition is reached.*

We have, from equation (20),

$$\frac{dt}{dx} = \frac{1}{\kappa(a-x)^2 - \frac{\lambda x^2}{4}}.$$

Since $t = 0$ when $x = 0$, this gives

$$t = \int_0^x \frac{dx}{\kappa(a-x)^2 - \frac{\lambda x^2}{4}}. \quad \text{..................(22)}$$

(Here we have written down the value of t at once as a definite integral, instead of first finding t as an indefinite integral with an undetermined constant added, and then determining the constant from the condition that $t = 0$ when $x = 0$. The value of t in equation (22), besides giving the proper value of $\frac{dt}{dx}$, obviously vanishes when $x = 0$.)

Now, write ρ^2 for $\frac{\lambda}{4\kappa}$, so that $\rho = \frac{1}{2}\sqrt{\frac{\lambda}{\kappa}}$, and we get from equation (22)

$$\kappa t = \int_0^x \frac{dx}{(a-x)^2 - \rho^2 x^2}. \quad \text{..................(23)}$$

* Lehfeldt's *Physical Chemistry.*

This integral can be evaluated by substitution in one of the general formulæ given on p. 262, but it is more instructive to work it out independently, by the method of partial fractions.

$$\text{Put } \frac{1}{(a-x)^2 - \rho^2x^2} = \frac{A}{a - x - \rho x} + \frac{B}{a - x + \rho x}.$$

To find A, multiply both sides by $a - x - \rho x$, and then put $a - x - \rho x = 0$, i.e. give x the value $a/(\rho + 1)$. The term in B vanishes, and we get,

$$A = \text{value of } \frac{1}{a - x + \rho x}, \text{ when } x = \frac{a}{\rho + 1}$$

$$= \frac{\rho + 1}{2a\rho}.$$

$$\text{Similarly, } B = \frac{\rho - 1}{2a\rho}.$$

Hence from equation (23),

$$\kappa t = \int_0^x \frac{\rho + 1}{2a\rho} \frac{dx}{a - (\rho + 1)x} + \int_0^x \frac{\rho - 1}{2a\rho} \frac{dx}{a + (\rho - 1)x}.$$

Now, at the start, $(a - x)^2 - \rho^2x^2$ has the value a^2, and the reaction goes on till it has the value 0. $\therefore$ $(a - x)^2 - \rho^2x^2$ never becomes negative, so that $a - (\rho + 1)x$ and $a + (\rho - 1)x$ are positive.

$$\text{Thus } \kappa t = \frac{1}{2a\rho}\Big| - \log(a - \overline{\rho + 1}x) + \log(a + \overline{\rho - 1}x)\Big|_0^x$$

$$= \frac{1}{2a} \log \frac{a + (\rho - 1)x}{a - (\rho + 1)x}.$$

(The student may easily verify by differentiation that this result leads to the value of $\frac{dt}{dx}$ given above equation (22); he will see at a glance that the result also gives $t = 0$ when $x = 0$.)

$$\text{Thus } t = \frac{1}{a\sqrt{\kappa\lambda}} \log_e \frac{a + (\rho - 1)x}{a - (\rho + 1)x}, \qquad \text{............(24)}$$

$$\text{where } \rho = \tfrac{1}{2}\sqrt{\frac{\lambda}{\kappa}}.$$

This equation gives the time required for x mols per litre of HI to be transformed into H_2 and I_2.

Numerical Calculation.

Some experiments by Bodenstein gave, at 440° C., the time being in seconds,

$$\kappa = 0{\cdot}00503,\ \lambda = 0{\cdot}365.$$

$$\therefore\ \rho = \tfrac{1}{2}\sqrt{\frac{\lambda}{\kappa}} = 4{\cdot}26.$$

Suppose we start with a litre of HI vapour at 440° C., and 760 mm. pressure.

1 litre of hydrogen at 0° C. and 760 mm. weighs 0·0899 gm.
∴ 1 litre of hydrogen at 440° C. and 760 mm. weighs $0{\cdot}0899 \times \frac{273}{713}$ gm.

The concentration of hydrogen, and therefore of HI also, at 440° C. and 760 mm., is thus $0{\cdot}0899 \times \frac{273}{713} \times \frac{1}{2}$ or 0·0172 mols per litre.

$$\therefore\ a = 0{\cdot}0172,$$

$$\text{and } \frac{1}{a\sqrt{\kappa\lambda}} = 1360.$$

The formula (24) now reduces to

$$t = 1360 \log_e \frac{0{\cdot}0172 + 3{\cdot}26x}{0{\cdot}0172 - 5{\cdot}26x}. \quad \ldots\ldots\ldots\ldots\ldots(25)$$

This is the equation we require connecting the time (t sec.) with the amount of HI decomposed (x mols per litre).

The equation gives a finite value of t for any value of x up to that value which makes the denominator on the right zero.

The final state (only reached, theoretically, after an infinite time) is given by the equation

$$0{\cdot}0172 - 5{\cdot}26x = 0,$$
$$\text{or } x = 0{\cdot}0033.$$

From this value of x, we get

$$\frac{x}{a} \times 100 = \frac{0{\cdot}0033}{0{\cdot}0172} \times 100 = 19,$$

so that when equilibrium is reached 19 per cent of the original HI has been decomposed. (Compare with the experimental figures for a slightly different temperature on p. 312.)

The time required for a given fraction of the HI, say 10 per cent, to decompose can be found readily from equation (25).

If $x = \frac{1}{10} \times 0{\cdot}0172$, we have

$$\begin{aligned} t &= 1360 \log_e \frac{1 + {\cdot}326}{1 - {\cdot}526} \\ &= 1360 \times 2{\cdot}303 \log_{10} \frac{1{\cdot}326}{0{\cdot}474} \\ &= 1400. \end{aligned}$$

The time is therefore 1400 sec. or 23 min. 20 sec.
Calculating similarly, we get the table:

Per cent Decomposition of HI.		Time.
0	....	0
5	....	10 min. 20 sec.
10	....	23 min. 20 sec.
15	....	44 min. 16 sec.
19	....	(equilibrium).

EXAMPLE

Trace the reaction in the opposite direction, starting from a mixture of hydrogen and iodine vapour in molecular proportions, and noting that the physics of the reaction in either direction is contained in the equation

$$\frac{d}{dt}(C_{HI}) = \lambda C_{H_2} C_{I_2} - \kappa (C_{HI})^2.$$

Prove that if a is the initial concentration of hydrogen, and therefore also of iodine, and if x is the amount of each transformed in time t,

$$2\frac{dx}{dt} = \lambda(a - x)^2 - 4\kappa x^2,$$

$$\text{whence } \frac{t}{2} = \int_0^x \frac{dx}{\lambda(a - x)^2 - 4\,\kappa x^2}. \quad \ldots\ldots\ldots\ldots\ldots\ldots(26)$$

$$\begin{aligned} \text{Taking } \lambda &= 0{\cdot}365 \\ \text{and } \kappa &= 0{\cdot}00503, \text{ at } 440^\circ \text{ C.,} \end{aligned}$$

express t as a function of x, and plot a curve of the amount of HI present as time progresses.

Find the "equilibrium" percentage of HI and compare it with the value previously obtained.

Exercise 15

1. Rutherford and Soddy found that the activity of Thorium X fell off with time in accordance with the following table:—

Time in Days from Start.		Activity of Th X.
2	....	100
3	....	88
4	....	72
6	....	53
9	....	29·5
10	....	25·2
13	....	15·2
15	....	11·1

The unit of "activity" is an arbitrary one, but the readings are proportional to the concentration of Th X.

Show that this reaction is a monomolecular one.

2. Cane sugar decomposes, in aqueous solution in the presence of a catalytic acid, into dextrose and laevulose, in accordance, quantitatively, with the equation

$$C_{12}H_{22}O_{11} + H_2O \rightarrow \underset{\text{dextrose.}}{C_6H_{12}O_6} + \underset{\text{laevulose.}}{C_6H_{12}O_6}.$$

The substances on the right and left respectively rotate a beam of polarized light in opposite directions, and hence, as the reaction proceeds, the observed rotation of the beam of light decreases. The reaction can therefore be studied with the polarimeter. The following are the results of an experiment (Lewis's *Physical Chemistry*):—

Time (min.).		Concentration.
0	....	0
30	....	1·001
60	....	1·946
90	....	2·770
130	....	3·726
180	....	4·676

The initial concentration of cane sugar was 10·023.

What conclusion do you draw as to the number of reacting molecules, and how do you reconcile your conclusion with the chemical equation?

3. An aqueous solution of dibromosuccinic acid transforms monomolecularly in accordance with the equation

$$C_4H_4O_4Br_2 \rightarrow C_4H_3O_4Br + HBr.$$

The reaction constant at 15° C. is 0·00000967; at 101° C., 0·0318, when time is measured in minutes.

Show that the time required to transform half the dibromosuccinic acid initially present is about seven weeks at 15° C., and less than half an hour at 101° C. (Van 't Hoff).

4. Phosphine decomposes quantitatively in accordance with the equation

$$4PH_3 \rightarrow P_4 + 6H_2.$$

An experiment by König gave

Time (hr.).	Pressure of Gaseous Mixture.
0	715·21 mm.
7·83	730·13 ,,
24·17	759·45 ,,
41·25	786·61 ,,
63·17	819·96 ,,
89·67	855·50 ,,

Examine these data with a view to settling whether the reaction is quadrimolecular, as the chemical equation suggests, or monomolecular. Write equations which you suggest represent the true course of the reaction.

5. It was shown by Fick that the "concentration gradient" is the quantity that determines the drift or diffusion of molecules in a solution. He found that the amount of the solute that would drift in unit time, normally across unit area, under a "concentration gradient" of unity, is a constant for a given temperature. This law is called *Fick's Law*, and the constant, the *diffusion constant*, D.

Assuming this law, show that the differential equation governing the change in concentration of a solution which arises from diffusion is

$$\frac{dc}{dt} = \mathrm{D}\frac{d^2c}{dx^2}, \quad \text{.......................(1)}$$

where c is the concentration and t the time, supposing the diffusion to take place in *one direction* x.

(a) Assuming D is constant, show that one solution of the equation is

$$c_0 + ae^{-\mathrm{D}n^2t} \sin nx,$$

where c_0, a, and n are constants.

(b) Find the values of n for which the concentration at $x = 0$ and $x = l$ is independent of t.

(c) Show that $c_0 + \sum_1^m a_n e^{-\mathrm{D}n^2t} \sin nx$ is also a solution of the equation, where Σ stands for the sum of all the sine terms got by putting $n = 1, 2, 3, \ldots m$.

Equation (1) also occurs in the theory of the conduction of heat. The solution given above is one of Fourier's.

CHAPTER XVI

Some Problems in Thermodynamics

"Economy of fuel is only one of the conditions a heat-engine must satisfy; in many cases it is only secondary, and must often give way to considerations of safety, strength, and wearing qualities of the machine, of smallness of space occupied, or of expense in erecting. To know how to appreciate justly in each case the considerations of convenience and economy, to be able to distinguish the essential from the accessory, to balance all fairly, and finally to arrive at the best result by the simplest means, such must be the principal talent of the man called on to direct and co-ordinate the work of his fellows for the attainment of a useful object of any kind."—SADI CARNOT, *Réflexions sur la Puissance Motrice du Feu* (1824).

There is no hard and fast line between the three conditions or *phases* in which it is convenient to regard matter as existing, namely, the *solid*, the *liquid*, and the *gaseous*. Thus, at ordinary temperatures, a lump of steel is solid; it can support a tensile stress, within limits, without any tendency to permanent change of shape. A liquid, such as water, cannot do this. A gas, such as hydrogen, shows no tendency to become liquid at ordinary temperatures, no matter how much pressure is applied to it. On the other hand, if the temperature of the hydrogen is very low, it tends to liquefy if the pressure is sufficiently increased, and in the so-called "critical" state, it appears to be very hard to decide whether the substance is definitely liquid or definitely gaseous.

Steam is another instance of the intermediate stage. It is difficult to determine whether a volume of "steam" is gaseous steam or atomized water.

A substance which is definitely gaseous under all conditions which we are considering, we shall call "a gas". Such gases are hydrogen, oxygen, nitrogen—so-called "permanent" gases—under ordinary conditions of temperature. Others, such as water, *when in the gaseous form*, we shall call *vapours*, thereby implying that the substance does not obey the simple gas equation $pv = R\theta$, and that liquefaction or even solidification may take place.

State.

In thermodynamics, we frequently refer to "the state of a body". This phrase denotes the physical and chemical condition of the body. To have a clear idea of the physical and chemical condition of a body we need information on the following points:—

(1) What is "the body" made of? Is it iron, hydrogen, &c.?
(2) Is the chemical nature of the body permanent? If it were hydrochloric acid vapour, for instance, it might change into hydrogen and chlorine.
(3) How much of it is there (its mass, as measured by its weight)?
(4) Where is it (its position in space)?
(5) Is it in the solid, liquid, or gaseous form, or partly one, partly another, as in a partially condensed vapour?
(6) What are its dimensions? What is its volume, if it is a gas?
(7) Is it at rest or in motion?

(8) Is it in a state of stress? What is its pressure if it is a gas?
(9) Is it magnetized?
(10) Is it electrified?
(11) How hot is it (its temperature)?
(12) Is it self-luminous, or sending out radiation of any known kind?

It is evident that, in general, a great deal of information is required in order to know the "state of a body".

We can greatly simplify the problem in certain cases:

(1) The body we are thinking about may be a definite amount of a specified gas, say nitrogen.
(2) We may be examining what happens to it in the gaseous state only. The equation $pv = R'\theta$ then holds very closely.
(3) It may not be subject to any electrical or magnetic influences, or to influences arising from radiation of any kind, so far as we can tell.
(4) Experiments show that ordinary nitrogen remains ordinary nitrogen over a very wide range of change of temperature and pressure.
(5) The body of gas considered may not alter its position appreciably relative to the earth, so that its potential energy, due to its weight, does not change appreciably.
(6) The body may be at rest as a whole.

These restrictions eliminate all the questions except (6), (8), and (11), and its "state" is therefore known when its dimensions (volume v), its state of stress (pressure p), and its temperature θ are specified.

We may therefore say that the state of *this* body (a *given* mass of a permanent gas) depends only on its pressure, volume, and temperature.

Functions of Two or More Variables.

In the preceding chapters we have dealt exclusively with functions of a single variable. In this chapter, we shall have to consider some simple functions of more than one variable.

The area of a rectangle is given by the product of its length and breadth.

If A stands for the area in square inches, l for the length and b for the breadth in inches,

$$A = lb.$$

The area depends on both the length and the breadth. These quantities have no relation to each other since we can vary either of them without *necessarily* affecting the other. They are said to be *independent* variables.

Keeping the length constant, if we vary the breadth, we vary the area; keeping the breadth constant, if we vary the length, we also vary the area.

If we vary the length *and* the breadth, we may or may not vary the area (e.g. halve the breadth, double the length, and we get no change in the area; but double both length and breadth and we get four times the area).

If a quantity w depends on the *independent* quantities x, y, z, &c., in such a way that changes in the values of any *one* of the quantities x, y, z, &c., are accompanied by changes in the value of w, w is said to be a function of x, y, z, &c., and in conformity with the notation for functions of one variable, we write

$$w = f(x, y, z, \&c.);$$

and if

$$w = f(x, y) = x^2 + y^2, \text{ say, then}$$
$$f(a, y) = a^2 + y^2,$$
$$\text{and } f(x, a) = x^2 + a^2,$$

and so on. (Compare p. 25.)

We can find the derivative of a function of several variables, with respect to any *one* of them, by the methods already given for a single variable; for in variations with respect to *one* of the variables, the others, being independent, are necessarily constant. So far, then, as differentiation with respect to any *one* variable is concerned, the function becomes one of a single variable.

For instance, if

$$w = f(x, y),$$

$$\text{then } \frac{df}{dx}^* = \underset{\Delta x \rightarrow 0}{\text{Lt}} \left[\frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}\right].$$

EXAMPLE

$$\text{Let } w = x^2 + y^2 = f(x, y).$$

Let the change in x be Δx.

y, of course, does not change as we are changing x, not y, and x and y are supposed to be *independent* variables.

$$\therefore \text{ change in } f = \{(x + \Delta x)^2 + y^2\} - (x^2 + y^2)$$
$$= (x + \Delta x)^2 - x^2$$
$$= 2x\Delta x + (\Delta x)^2.$$
$$\frac{\text{change in } f}{\text{change in } x} = 2x + \Delta x.$$
$$\therefore \underset{\Delta x \rightarrow 0}{\text{Lt}} \left(\frac{\text{change in } f}{\text{change in } x}\right) = \underset{\Delta x \rightarrow 0}{\text{Lt}} (2x + \Delta x)$$
$$= 2x.$$
$$\therefore \frac{df}{dx} = 2x.$$

We should obtain this result if we applied our ordinary rules for a single variable to $x^2 + y^2$,

regarding y as constant, i.e. regarding $x^2 + y^2$ as a function of x only.

Similarly,
$$\frac{df}{dy} = 2y.$$

A function of two variables has thus two first derivatives, one with regard to *each* variable.

* $\frac{df}{dx}$ is an abbreviation for $\frac{d}{dx} f(x, y)$.

A function of n variables has n first derivatives. A special notation is usually used for these several first derivatives.

Thus, if

$$w = f(x, y),$$

we write

$$\frac{\partial f}{\partial x} \text{ instead of } \frac{df}{dx}, \text{ and}$$

$$\frac{\partial f}{\partial y} \quad \text{,,} \quad \frac{df}{dy}.$$

The derivatives are called *partial* derivatives. Thus $\frac{\partial f}{\partial x}$ is called the first partial derivative of $f(x, y)$ with respect to x, and $\frac{\partial f}{\partial y}$, the first partial derivative of $f(x, y)$ with respect to y.

Characteristic Equation.

It is found experimentally that there is a definite relation between the pressure, volume, and temperature of a given mass (say 1 lb.) of any substance. This relation takes its simplest form for a gas. It is of the type

$$pv = R'\theta, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

where p is pressure, v volume, θ temperature, and R' is a constant for a particular gas, but is different for different gases.

If we write $R' = \frac{R}{m}$, where m is the molecular weight of the gas in question, R is a constant for all gases.

Its value depends on the units we use for p, v, and θ.

Formula (1) becomes

$$pv = \frac{R}{m}\theta, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

where p is the pressure per unit area, v the volume per unit mass, θ the absolute temperature, m the molecular weight of the gas, and R the gas constant. The value of R is given in the following table:—

Quantity.	British.	Scientific.
p	Pounds per square *foot.*	Dynes per square centimetre.
v	Cubic feet per pound.	Cubic centimetres per gramme.
θ	$(461 + t)$, t being in degrees F.	$(273 + t)$, t being in degrees C.
m	Molecular weight.	Molecular weight.
R	15·45 (foot-pounds per pound mol per degree F. abs.). (O = 16.)	$83\cdot15 \times 10^6$ (ergs per gramme mol per degree C. abs.). (O = 16.)

EXAMPLES

1. Find the "gas equation" for air in Engineer's Units.

For oxygen let $\frac{pv}{\theta} = R_1$, for air $\frac{pv}{\theta} = R_2$.

Suppose standard p and θ taken in both these equations. Then, obviously,

$$\frac{R_2}{R_1} = \frac{\text{volume of air}}{\text{volume of oxygen}} = \frac{\text{density of oxygen}}{\text{density of air}}.$$

$$\therefore R_2 = R_1 \times \frac{\text{density of oxygen}}{\text{density of air}}$$

$$= \frac{R}{32} \times \frac{1}{\cdot 9047} = \frac{1545}{32 \times \cdot 9047}$$

$$= 53\cdot 4.$$

$$\therefore pv = 53\cdot 4\theta, \text{ for air,}$$

where p is in pounds per square *foot*, v in cubic feet per pound, and θ in degrees F. absolute.

2. Find the "gas equation" for nitrogen in scientific units.

m for nitrogen is 28.

$$\therefore \; R_N = \frac{83 \cdot 15 \times 10^6}{28} = 2 \cdot 97 \times 10^6.$$

$$\therefore \; pv = 2 \cdot 97 \times 10^6 \theta,$$

where p is in dynes per square centimetre, v in cubic centimetres per gramme, and θ in degrees C. absolute.

For steam, the characteristic equation is much more complicated. Callendar's equation is

$$V = 85 \cdot 65 \frac{\theta}{p} - 0 \cdot 4213 \left(\frac{671 \cdot 6}{\theta}\right)^{\frac{10}{3}} + 0 \cdot 016,$$

where V is in cubic feet per pound, θ in degrees F. absolute, and p in pounds per square *foot*. This equation holds for dry or superheated steam.

To find the *work done by* a gas, expanding slowly and isothermally.

By p. 151 the work done is

$$\int_{v_0}^{v_1} p\,dv,$$

where p stands for the pressure in pounds per square *foot* and v for the volume in cubic feet. The work done is then measured in foot-pound units.

For *isothermal* expansion, the characteristic equation gives

$$pv = R'\theta = \text{a constant, since } \theta \text{ is constant.}$$

$$\therefore \; p = \frac{R'\theta}{v}.$$

$$\therefore \; \int_{v_0}^{v_1} p\,dv = R'\theta \int_{v_0}^{v_1} \frac{dv}{v} = R'\theta \log \frac{v_1}{v_0} = R'\theta \log_e r, \quad (3)$$

where r is the ratio of expansion, i.e. $\frac{v_1}{v_0}$.

The work done in expansion obviously depends on the amount of gas expanding.

It is usual, in engineering, to take a standard amount of gas, 1 lb. For other amounts, the work done in expansion is proportional to the mass.

Equation (3) therefore becomes, *for air*,

$$\text{Work done, } w = 53{\cdot}4\ \theta \log_e(v_1/v_0) \text{ per lb.} \ldots(4)$$

For *isothermal* expansion, the work done is therefore proportional to the logarithm of the ratio of expansion r $(= v_1/v_0)$.

The graph of $w = 53{\cdot}4\ \theta \log r$ is the logarithmic curve (fig. 2, p. 239). When $r = 1$, $w = 0$, and when $r < 1$, w is negative, i.e. the gas is being compressed by an outside agency, and work is being done *on* it, i.e. energy is being added to it.

EXAMPLE

Assume air at 200° F. expands slowly and isothermally from an initial volume v_0 to a final volume of 1 c. ft. at atmospheric pressure. Let the ratio of expansion be 6. What is the initial pressure of the air, and how much work is done in the expansion? (Take 13 c. ft. of air at atmospheric pressure and 60° F. as weighing 1 lb.)

1 c. ft. of air at 200° F. and atmospheric pressure weighs

$$\tfrac{1}{13} \times \tfrac{521}{661} = {\cdot}0606 \text{ lb.} \quad \text{by the gas}$$

The initial pressure is given by $p_0 \times \frac{1}{6} = 1 \times 1.$ equation.

$$\therefore p_0 = 6 \text{ atmospheres}$$
$$= 88 \text{ lb./square inch.}$$
$$\log_e 6 = 1{\cdot}7918.$$
$$\therefore \text{Work, } \textit{per lb.} = 53{\cdot}4 \times 661 \times 1{\cdot}7918 \text{ ft.-lb.}$$
$$\therefore \text{Actual work done} = 0{\cdot}0606 \times 53{\cdot}4 \times 661 \times 1{\cdot}7918 \text{ ft.-lb.}$$
$$= 3833 \text{ ft.-lb.}$$

Intrinsic (or Internal) Energy.

It is well known that matter, especially of certain kinds, possesses large amounts of stored energy, which we can tap by suitable treatment.

One ounce of gunpowder, for instance, may give out 3 or 4 foot-tons of energy if exploded by a spark.

This energy cannot come from the spark. The energy required to produce the spark can be easily measured electrically, and is quite trifling in comparison with the energy given out by the gunpowder. The energy, which shows itself when the stimulus of the spark is applied, must therefore have been stored in the powder.

On the other hand, matter can be made *to absorb* energy, as when water is heated.

These facts leads us to suspect that every portion of matter has a definite amount of energy associated with it. This energy may be called the *Internal* or *Intrinsic* energy of the body or system of bodies. In some cases we can add to or take from the store by suitable processes, e.g. by sparking the gunpowder, or by heating the water.

It is interesting to consider how this energy may be stored in the body. We will consider a certain volume of a gas. By the molecular theory, this volume of gas consists of a multitude of molecules confined within the walls of the containing vessel, and all in a state of violent agitation.

Each of the molecules has a perceptible mass and a velocity, and therefore each possesses both kinetic energy and momentum.

Each, again, might attract the others as do the stars and planets.

Again, each molecule may itself be a minute system of some kind, possessing an amount of energy proper to itself.

It is evident that in a gas the intrinsic energy may consist of:

(1) Kinetic energy of molecular motion.
(2) Strain energy due to the forces between the molecules.
(3) Energy locked up in each molecule.

We may regard the intrinsic energy of a body as the energy associated solely with the matter within the boundary of the body. This capacity which the body possesses for doing work must, as we shall see later, arise from, and therefore be dependent upon, its actual physical and chemical state.

A Cycle.

A series of physical or chemical changes which a body may undergo is called a "cycle" if the final "state" of the body is exactly the same as the initial "state".

Joule's Equivalent.

It is common knowledge that heat can be produced by expending mechanical energy. A piece of iron gets hot when hammered on an anvil. Joule found that to produce 1 B.Th.U. of heat, 772 ft.-lb. of mechanical work must be supplied either from the internal energy of the body or from external sources, and this number is quite independent of the initial state of the body, or of the manner in which the mechanical energy is supplied.

The number now used, 778, is called Joule's *Mechanical Equivalent of Heat.*

Extension of Joule's Discovery.

After this discovery was made, it was soon recognized that *all* forms of energy are "equivalent", each pair of forms having an appropriate conversion factor. For instance, in electrical engineering,

1 Kelvin, or K.W.H. = 3412 B.Th.U.'s.

The forms of energy which have so far been recognized are*:—

1. Kinetic energy.
2. Potential ,,
3. Heat ,,
4. Strain ,,
   (1 and 4 are also combined in sound.)
5. Light and radiant energy.
6. Electric energy.
7. Magnetic ,,
8. Chemical ,,

Conservation of Energy.

Then followed the *principle* that the total amount of energy of all kinds in an isolated body or system of bodies is constant, and that it is only *transformation*, not creation nor destruction, that is possible.

By "isolated body or system of bodies" is meant a system screened from all physical or chemical action arising from bodies not in the system under consideration. The system may be imagined to be enclosed in an "adiabatic" shell, through which forces cannot act nor energy penetrate (*α*, not; *διά*, through; *βαίνειν*, to go).

* Poynting & Thomson's *Heat*, p. 155.

First Law of Thermodynamics.

Consider such an isolated system consisting of a hot body H, a non-conducting cylinder C, a frictionless piston P, a compressed spring S, and a gas

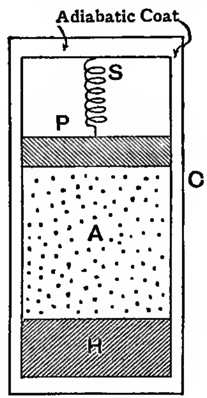


Fig. 1

A inside the cylinder. Suppose the piston is held in position by a trigger.

Consider A as one body and the rest of the apparatus as another body B.

Let the trigger be released and we will suppose that the trifling amount of energy required to release the trigger is included in the total energy of the system at the moment of release.

When equilibrium is restored:

(1) The spring has compressed the gas and thereby done work *on* it. Suppose this work is δW, positive when work is done *on* the body A.

(2) The hot body H has warmed the gas, and therefore pumped heat energy into it. Suppose δQ

is the heat energy pumped *into* A, positive when going *into* A.

A has clearly gained energy.

It can only have absorbed this energy by having its internal energy increased. Let E be the excess of the body's internal energy over its internal energy in a chosen standard state. Let E increase by δE.

$$\therefore\ \delta E = \delta Q + \frac{\delta W}{778}, \qquad \text{..............(5)}$$

where δE and δQ are measured in heat units and δW in units of *work*. If we measure δW in *heat units*, putting 778 ft.-lb. $\equiv$ 1 B.Th.U., the factor $\frac{1}{778}$ can be omitted.

Equation (5) is a direct result of the principle of *the conservation of energy*.

The internal energy of a body depends on its physical state. The truth of this statement can be seen by considering the consequences of its denial. If it were not true, then we could pump heat energy into A, by heating it, without affecting its physical or chemical state. This cannot be done with any natural body. The addition or subtraction of energy always affects in some way the physical or chemical state of the body.

Once the numbers specifying the state of the body are set, the internal energy is a unique and definite amount, depending only on these numbers.

This experimental fact, namely, that the internal energy is a function of the state of the body, is known as the First Law of Thermodynamics.

If then E = ϕ (x, y, z, w, &c.), where x, y, z, w, &c., are numbers (co-ordinates) which define the state of the body, and we denote the state of the body before the trigger is released by x_1, y_1, z_1, w_1, &c., and, after, by x_2, y_2, z_2, w_2, &c., then

$$\begin{aligned} E_2 - E_1 &= \phi(x_2, y_2, z_2, w_2, \ldots) - \phi(x_1, y_1, z_1, w_1, \ldots) \\ &= \int_{E_1}^{E_2} dE \\ &= \int\left(dQ + \frac{dW}{778}\right). \end{aligned}$$

If the initial state be described briefly as "the state A", and the final state as "the state B", we may conveniently write the latter integral in the form

$$\int_A^B \left(dQ + \frac{dW}{778}\right).$$

Now, suppose the body undergoes *a cyclic change.* The total change in E must be zero, since by hypothesis the initial and final states of the body are identical.

$$\therefore \left(\int\right)\left(dQ + \frac{dW}{778}\right) = 0, \ldots\ldots\ldots\ldots(6)$$

where $\left(\int\right)$ means integration carried out for a cyclic change.

We shall suppose that mechanical energy is measured in heat units. We can then ignore 778 in the equations and put

$$E_B - E_A = \int_A^B \left(dQ + dW\right), \ldots\ldots\ldots\ldots(7)$$

and E is a function of the numbers that define the state of the body.

Since the internal energy is unchanged in any cyclic process, any net work which the substance going through the cycle—the "working" substance

—may do can only be done at the expense of the net heat received by the body from outside sources. That is to say, the body acts merely as a vehicle of energy, its own energy not being permanently altered.

In fact, we have

$$\left(\int\right)(dQ + dW) = 0, \text{ by equation (6),}$$

$$\text{hence } \left(\int\right)dQ = -\left(\int\right)dW,$$

i.e. the heat received by the body in the cycle = work done *by* the body on the outside world during the cycle.

(Notice the sign of $\left(\int\right)dW$. Work done *on* the body in question is positive, so work done *by* the body is negative.)

For this reason, it is not surprising to find that the thermodynamic properties of cycles are in some important respects independent of the nature of the working substance, even though some materials are good vehicles, in that they can convey large amounts of energy per pound, while others are bad ones.

Sometimes the principle of the conservation of energy is regarded as the First Law of Thermodynamics. The facts may be put as follows:—

1. In no natural process can energy be created or destroyed. If x units of energy of kind A disappear (appear) then an exactly equivalent amount of energy of some other kind appears (disappears). Energy of every kind can therefore be expressed in any unit of energy, say the foot-pound or the B.Th.U. or the Board of Trade unit of electrical energy, and the total amount of energy in any isolated system is constant. This principle leads to

$$\delta E = \delta Q + \delta W,$$

where E, Q, and W are measured in terms of the same unit of energy.

2. The internal energy of a body depends on its state. It is impossible to add energy to a body or subtract energy from it without affecting its state. This may be regarded as the First Law of Thermodynamics.

It leads to the equations

$$E = \text{a function of the state of body, } \ldots\ldots\ldots\ldots\ldots (8)$$

$$\text{and } (\int) dE = 0.$$

3. It will be seen later that the body possesses another property—its entropy—which depends solely on its state, and which cannot be altered without affecting its state. The Second Law of Thermodynamics leads directly to this conclusion. If ϕ is the measure of this property,

$$\phi = \text{a function of the state of the body. } \ldots\ldots\ldots\ldots (9)$$

Reversible Changes.

If a body A at temperature $\theta°$ F. is in contact with a body B at the same temperature,

A $\theta°$	B $\theta°$

no exchange in heat energy will take place.

If the temperature of A drops a little, relative to that of B, a flow of heat will be set up from B to A; while if it rises a little, relative to B, the flow of heat energy will be reversed.

The essential feature of a flow of heat energy between two bodies A and B under reversible conditions is that

$$\theta_B = \theta_A + \epsilon \text{ if heat flows from B to A,}$$
$$\theta_B = \theta_A - \epsilon \quad ,, \quad ,, \quad ,, \quad \text{A} \;,, \; \text{B,}$$

where ϵ is any very small positive number. Provided ϵ is not zero, exchange of heat energy will take place, but if ϵ is infinitesimally small, the exchange of heat energy will take place at an infinitesimal rate.

An exchange of mechanical energy can take place under similar conditions.

Let C (fig. 2) be a non-conducting cylinder, P a frictionless piston. Let there be air, for instance, behind the piston at pressure p lb. per square inch.

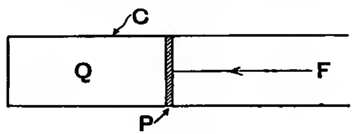


Fig. 2

Let A be the area of the piston in square inches. Suppose a unidirectional force F equal to pA in amount is applied to prevent the piston moving outwards.

So long as $pA = F$ no movement of P takes place.

If $pA = F + \epsilon$ where ϵ is a very small positive number, the piston moves *extremely slowly* outwards.

If $pA = F - \epsilon$, the movement is reversed.

These conditions are essentially similar to those which determine reversible heat exchange.

When $pA = F + \epsilon$ and the piston moves outwards, the body Q, i.e. the gas behind P, loses energy equal to $F \times l$, where l is the distance the piston moves outwards.

When $pA = F - \epsilon$, the gas gains energy by having work done on it by external agencies.

A Reversible Cycle.

A reversible cycle is one in which every exchange of energy is reversible.

Adiabatic Expansion.

When a gas or vapour expands adiabatically—i.e. without receiving or giving up heat to other bodies—and reversibly, it does work and must give up part of its internal energy in the process. Experiments show that the expansion takes place in accordance with an equation of the type:

$$pv^n = \text{constant.} \qquad \ldots\ldots\ldots\ldots\ldots(10)$$

$$\text{In air } n = \gamma = \frac{C_p}{C_v} = \begin{cases}\text{ratio of specific heats at} \\ \text{constant pressure and} \\ \text{at constant volume}\end{cases}$$

$$= 1{\cdot}400 \text{ about.}$$

$$\text{For initially dry steam } n = 1{\cdot}135 \text{ about.}$$

When we are dealing with gases, this equation, pv^n = a constant, enables us to express any one of the three quantities, pressure (p), volume (v), and temperature (θ), in terms of the other two.

For example, if we wish to have the equation for θ and v we note that

$$\frac{pv}{\theta} = R',$$

and $pv\,v^{n-1} = \kappa$ by (10), where κ is a constant.

$$\therefore\ R'\theta\, v^{n-1} = \kappa$$

$$\therefore\ \theta\, v^{n-1} = \text{constant.} \qquad \ldots\ldots\ldots\ldots\ldots(11)$$

Thermodynamics of a Gas.

The three quantities which completely define the state of a given mass of a gas are its pressure, its volume, and its temperature.

We have given reasons for thinking that the internal energy of a substance depends only on its state. Hence for a gas

E is a function of v and θ,

since p is known when v and θ are given, from the equation

$$pv = R'\theta.$$

Joule and Kelvin have shown experimentally that if a given mass of gas is allowed to expand without an exchange of heat or mechanical work, *its temperature does not alter.*

Consider a small communication of energy to the given mass of gas.

We have by equation (5)

$$\delta E = \delta Q + \delta W, \quad \ldots\ldots\ldots\ldots\ldots\ldots (12)$$

where δQ is the heat energy put *into* the gas in B.Th.U.'s;

δW, the mechanical energy put *into* the gas in B.Th.U.'s;

and δE, the increase in the internal energy of the gas in B.Th.U.'s.

In Joule and Kelvin's experiment,

$\delta Q = 0$, there being no exchange of heat;

$\delta W = 0$, there being no exchange of mechanical energy;

therefore $\delta E = 0$, by equation (12).

Now, in all bodies, E may be regarded as a function of v and θ only, so that we may write here

$$E = f(v, \theta).$$

If θ remains constant, as it did in the experiments of Joule and Kelvin, any increase of E depends on the increase of v, and we have

$$\delta E = \frac{\partial f}{\partial v}\delta v, \text{ approximately (p. 137).}$$

Since δE is 0 in the experiments, and δv is not 0, we must have $\frac{\partial f}{\partial v} = 0$, for all values of v.

$\therefore f(v, \theta)$ cannot depend on v, for, if it did, it would change with v and therefore have a derivative $\frac{\partial f}{\partial v}$, which would not be identically zero.

$$\therefore E = f(\theta), \text{ a function of } \theta \text{ only.}$$

It is thus an experimental fact that, for a "permanent" gas, E depends only on the temperature.

The function $f(\theta)$ can be determined as follows:

Let a change take place at constant volume. Then the gas does no mechanical work.

$$\therefore \delta E = \delta Q.$$

$$\text{But } \delta Q = C_v \delta\theta,$$

where C_v is the specific heat of the gas at constant volume.

$$\therefore \delta E = C_v\delta\theta, \text{ or } \frac{dE}{d\theta} = C_v.$$

C_v is therefore a function of θ only, and we have

$$E = \int C_v d\theta + \text{constant.}$$

By experiment it is found that C_v is practically constant over a wide range of temperature, so that this equation becomes

$$E = C_v\theta + \text{constant.} \quad \ldots\ldots\ldots\ldots(13)$$

This is the function of θ required.

Reverting now to a *reversible* change with expansion, in which the working substance does work $p\delta v$, we get

$$\delta E = \delta Q - p\delta v,$$

$$\text{i.e. } \delta Q = C_v\delta\theta + p\delta v.$$

$$\therefore \frac{\delta Q}{\theta} = C_v\frac{\delta\theta}{\theta} + \frac{p\delta v}{\theta}.$$

$$\text{But } \frac{p}{\theta} = \frac{R'}{v}, \text{ by (1).}$$

$$\therefore \frac{\delta Q}{\theta} = C_v\frac{\delta\theta}{\theta} + R'\frac{\delta v}{v}$$

$$= C_v\delta(\log_e\theta) + R'\delta(\log_e v), \text{ (p. 243)}$$

$$= \delta\{C_v \log_e\theta + R' \log_e v\}$$

$$= \delta\phi,$$

if we introduce a function ϕ defined by the equation

$$\phi = C_v \log_e\theta + R' \log_e v + \text{constant.} \quad \text{.........(14)}$$

The function ϕ thus defined is a function of θ and v—i.e. of *the state of the gas*—and is just as definitely a property of the gas in a given state as E is.

The function ϕ is calculable, except for the additive constant, if θ and v are given.

If we measure ϕ from a definite selected state of the body, so that $\phi = 0$ when $v = v_0$ and $\theta = \theta_0$, then

$$0 = C_v \log_e\theta_0 + R' \log_e v_0 + \text{constant.}$$

$$\therefore \text{constant} = -C_v \log_e\theta_0 - R' \log_e v_0,$$

$$\text{and } \phi = C_v \log_e\frac{\theta}{\theta_0} + R' \log_e\frac{v}{v_0} = \psi(v, \theta). \text{(15)}$$

Since $\phi = \psi(v, \theta)$, ϕ must return to its original value after going round any cycle, starting from the state v, θ, and returning to the same state v, θ, i.e.

$$(\int) d\phi = 0.$$

$$\text{But } \frac{\delta Q}{\theta} = \delta\phi \text{ (p. 340);}$$

$$\text{hence } (\int)\frac{dQ}{\theta} = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots (16)$$

The function ϕ is called the *entropy* of the gas, and is reckoned per pound of the gas from $\phi = 0$ at normal temperature and pressure.

It should also be carefully noted that all the changes discussed so far are *reversible* ones, i.e. the temperatures of the bodies exchanging heat are equal ($T = \theta$), and the driving and resisting mechanical forces are balanced. (See p. 335.)

Practical Engine Cycles.

All engine cycles are irreversible both as regards mechanical and thermal exchanges; otherwise the cycle would be carried out indefinitely slowly. In expansion of the working substance, for instance, the friction of the piston must be overcome, and this is an irreversible resistance. Also, the working substance is heated and cooled very rapidly, therefore $T > \theta$ when heating up and $T < \theta$ when cooling, *by a considerable amount.* The exchanges of heat are therefore irreversible. Further, in the case of gas- or oil-engines, the process which goes on in the cylinder is not even a "cycle" as defined on p. 329; for the working substance burns irreversibly in the process, and therefore its initial and final "state" cannot possibly be the same. The best we can do, theoretically, is to construct a "cycle"—reversible if possible—which resembles, as closely as may be, the actual process that goes on in the cylinder or engine.

The two most important internal combustion engine cycles are the Otto and Diesel cycles, though these cycles are not truly cycles in the strict thermodynamic sense.

Otto Cycle.

1. A combustible mixture of gas and air is sucked into a cylinder at approximately atmospheric pressure.

2. It is compressed, approximately adiabatically, to a small volume at a high pressure and a fairly high temperature.

3. It is exploded and the pressure rises more or less abruptly according to the speed of the explosion.

4. The high-pressure mixture (no longer the same, chemically, as it was in stages 1, 2, and 3) expands approximately adiabatically, doing external work and getting cooler in the process.

5. The burnt mixture at a low pressure and a fairly low temperature is expelled from the cylinder into the atmosphere.

In this cycle, we get one explosion per two revolutions of the engine, and the engine is known as a four-stroke engine, i.e. four strokes go to each cycle. The programme is repeated in the next two revolutions of the engine, and so on indefinitely. The nearest ideal cycle to this one is as follows:—

Start with a volume v_1 of any permanent gas at pressure p_1, and temperature θ_1, in an ideal cylinder. The suffixes refer to the states at the corresponding points on the diagrams. Trace out the following changes on the p, v diagram (fig. 3):

1. Heat the gas at constant volume by applying hot bodies at progressively increasing temperatures (reversible) from θ_1 to θ_2. Its pressure will be increased.

2. Expand the high-pressure gas adiabatically and reversibly against a resisting pressure $F = p$ (reversible). The gas does work, and therefore its temperature drops to θ_3. Let p_3 and v_3 be the pressure and volume at the end of the expansion.

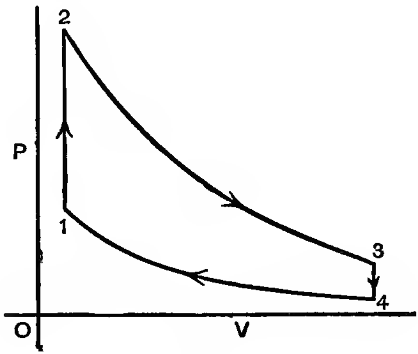


Fig. 3

3. Cool the gas at constant volume v_3 from θ_3 to θ_4, by applying cold bodies of progressively decreasing temperatures (reversible). The cooling is stopped at the temperature θ_4, which is such that the point 4 is on the adiabatic curve through the point 1.

4. Compress the gas adiabatically from p_4, v_4, θ_4 to p_1, v_1, θ_1. This is possible since the point 4 was taken on the adiabatic through the point 1. The

temperature rises, of course, during this compression, as external work is done on the gas and heat does not enter or leave it.

This cycle yields a definite amount of external work which is absorbed in a mechanical form. The p, v diagram shows a definite area which measures this available work. Now, the efficiency of the cycle is the ratio

$$\frac{\text{number of units of work done}}{\text{mechanical equivalent of heat taken in from hot bodies}},$$

and it is very important to know what this fraction (η) is, as it measures the efficiency of the cycle as a

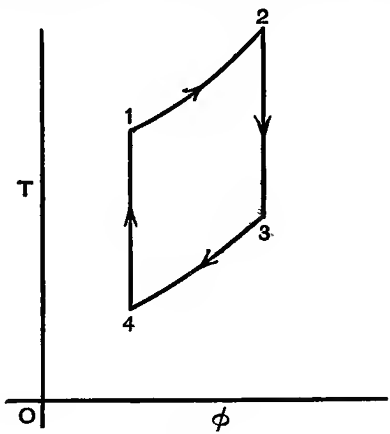


Fig. 4

means of transforming heat energy into mechanical energy.

The numerator we know from the p, v diagram and the formula

$$W = (\int) p\,dv. \quad \text{(See p. 153.)}$$

We can calculate the denominator by using the fact that

$$\left(\int\right)d\phi = 0,$$

where ϕ is the entropy per pound of gas.

Trace the change in entropy as the gas goes through the cycle beginning at state 1. (See fig. 4.)

Stage.	Heat into Gas.	Increase of Entropy.
1	$C_v(\theta_2 - \theta_1)$	$C_v\int_{\theta_1}^{\theta_2}\frac{d\theta}{\theta} = C_v \log_e\frac{\theta_2}{\theta_1}$
2	0	0
3	$-C_v(\theta_3 - \theta_4)$	$-C_v \log_e\frac{\theta_3}{\theta_4}$
4	0	0

$$\therefore \eta = \frac{C_v(\theta_2 - \theta_1 - \theta_3 + \theta_4)}{C_v(\theta_2 - \theta_1)} = 1 - \frac{\theta_3 - \theta_4}{\theta_2 - \theta_1},$$

$$\text{also}\left(\int\right)d\phi = 0 = C_v \log_e\frac{\theta_2}{\theta_1} - C_v \log_e\frac{\theta_3}{\theta_4}.$$

$$\therefore \frac{\theta_2}{\theta_1} = \frac{\theta_3}{\theta_4}.$$

$$\therefore \frac{\theta_2}{\theta_1} - 1 = \frac{\theta_3}{\theta_4} - 1.$$

$$\therefore \frac{\theta_2 - \theta_1}{\theta_1} = \frac{\theta_3 - \theta_4}{\theta_4}.$$

$$\therefore \frac{\theta_3 - \theta_4}{\theta_2 - \theta_1} = \frac{\theta_4}{\theta_1} = \frac{\theta_3}{\theta_2}.$$

$$\therefore \eta = 1 - \frac{\theta_3 - \theta_4}{\theta_2 - \theta_1} = 1 - \frac{\theta_3}{\theta_2} = \frac{\theta_2 - \theta_3}{\theta_2} = \frac{\theta_1 - \theta_4}{\theta_1}.$$

Now, in fig. 3, the curved boundary lines of the cycle are adiabatic lines, i.e.

$$\theta v^{\gamma-1} = \text{constant (p. 337).}$$

$$\therefore \frac{\theta_4}{\theta_1} = \left(\frac{v_1}{v_4}\right)^{\gamma-1} = \left(\frac{1}{r}\right)^{\gamma-1},$$

$$\text{where } r = \frac{v_4}{v_1} = \text{ratio of expansion,}$$

$$\text{and } \eta = 1 - \left(\frac{1}{r}\right)^{\gamma-1}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(17)$$

This is the formula for the efficiency of the ideal gas cycle which most nearly corresponds to the actual Otto cycle. The nature of the gas does not affect the result. It is the same for all gases which obey the gas law $pv = R'\theta$.

Oil-engines—Diesel Cycle.

1. A charge of air is sucked into a cylinder at approximately atmospheric pressure and temperature.

2. It is highly compressed, more or less adiabatically, so that its temperature is sufficiently high for oil to burn when squirted into it.

3. By means of a pump, oil is squirted into the cylinder and burns. While burning proceeds, expansion is allowed to take place so that the pressure does not rise, i.e. the heat energy of combustion, except such as is used in the expansion referred to, goes into the gas at constant pressure.

4. When combustion is complete, the burnt mixture is allowed to expand more or less adiabatically, thereby doing work and getting cooler.

5. When the piston gets to the end of its stroke, the cylinder is connected to the atmosphere, and the con-

tents of the cylinder, at a fairly low pressure and temperature, are expelled from the cylinder into the atmosphere during the ensuing stroke.

The ideal gas cycle nearest to this one is as follows:

Imagine a volume of gas, initially at pressure p_1,

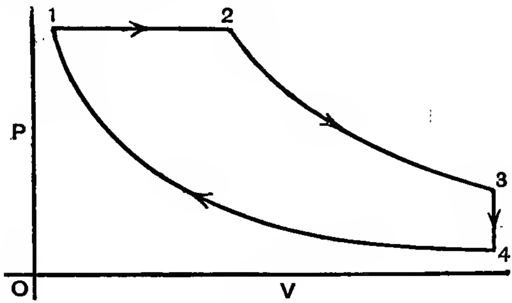


Fig. 5

volume v_1, and temperature θ_1, in an ideal cylinder, and subject it to the following cycle (see fig. 5):—

1. Heat it at constant pressure, by means of a set of hot bodies at progressively increasing temperatures, from p_1, v_1, θ_1 to p_2, v_2, θ_2 ($p_2 = p_1$).

2. Expand it adiabatically to p_3, v_3, θ_3.

3. Cool the gas at constant volume, by applying cold bodies at progressively decreasing temperatures, from p_3, v_3, θ_3 to p_4, v_4, θ_4 ($v_3 = v_4$).

4. Compress the gas adiabatically to its initial state, the point 4 being chosen upon the adiabatic line through 1.

The T, ϕ diagram is shown in fig. 6.

The cycle is reversible, hence $(\int)d\phi = 0$.

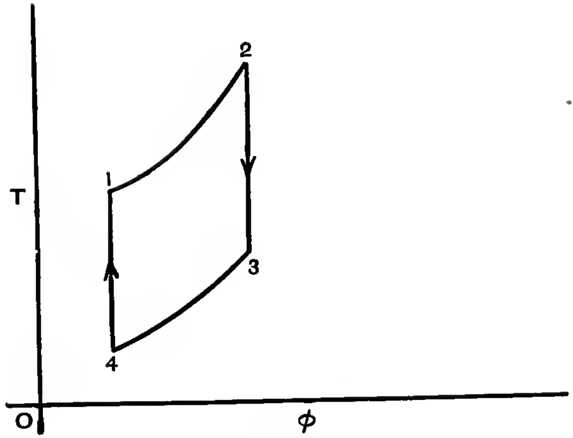


Fig. 6

The changes of entropy are given in the following table :—

Stage.	Heat into Gas.	Increase of Entropy.
1	$C_p(\theta_2 - \theta_1)$	$C_p \log_e \frac{\theta_2}{\theta_1}$
2	0	0
3	$- C_v(\theta_3 - \theta_4)$	$- C_v \log_e \frac{\theta_3}{\theta_4}$
4	0	0

$$\left(\int\right) d\phi = 0 \text{ gives}$$

$$C_p \log \frac{\theta_2}{\theta_1} = C_v \log \frac{\theta_3}{\theta_4},$$

$$\gamma C_v \log \frac{\theta_2}{\theta_1} = C_v \log \frac{\theta_3}{\theta_4}, \text{ where } \gamma = \frac{C_p}{C_v}.$$

$$\therefore\ \gamma \log\frac{\theta_2}{\theta_1} = \log\frac{\theta_3}{\theta_4}.$$

$$\therefore\ \left(\frac{\theta_2}{\theta_1}\right)^\gamma = \frac{\theta_3}{\theta_4}.$$

The efficiency of the cycle is given by

$$\eta = \frac{C_p(\theta_2 - \theta_1) - C_v(\theta_3 - \theta_4)}{C_p(\theta_2 - \theta_1)}$$

$$= 1 - \frac{1}{\gamma}\frac{\theta_3 - \theta_4}{\theta_2 - \theta_1}$$

$$= 1 - \frac{1}{\gamma}\left\{\frac{\left(\frac{\theta_2}{\theta_1}\right)^\gamma - 1}{\frac{\theta_2}{\theta_1} - 1}\right\}\frac{\theta_4}{\theta_1}.$$

But $\frac{p_1 v_1}{\theta_1} = \frac{p_2 v_2}{\theta_2}$.

$$\therefore\ \frac{\theta_2}{\theta_1} = \frac{v_2}{v_1},\ \text{as } p_2 = p_1$$

$$= \rho,\ \text{say},$$

and θ_4 and θ_1 are values of θ on an adiabatic line.

$$\therefore\ \frac{\theta_4}{\theta_1} = \left(\frac{1}{r}\right)^{\gamma-1},$$

as proved in connection with equation (17).

$$\therefore\ \eta = 1 - \frac{1}{\gamma}\left\{\frac{\rho^\gamma - 1}{\rho - 1}\right\}\left(\frac{1}{r}\right)^{\gamma-1}. \quad \ldots(18)$$

This formula gives the value of the efficiency required, where r is the expansion ratio v_4/v_1 and ρ is the expansion ratio at constant pressure, v_2/v_1.

Since $\rho > 1$, $\rho^\gamma - 1 > \gamma(\rho - 1)$.

This inequality is obviously equivalent to

$$(\gamma - 1) > (\gamma\rho - \rho^{\gamma}).$$

Now $(\gamma - 1)$ is the value of the right-hand expression when $\rho = 1$.

If, therefore, we put $f(\rho)$ for $(\gamma\rho - \rho^{\gamma})$, then

$$(\gamma - 1) = f(1).$$

Now, $\frac{d}{d\rho}f(\rho) = \gamma(1 - \rho^{\gamma-1})$, which is negative for all values of $\rho > 1$. Hence $f(\rho)$ decreases as ρ increases from 1.

$$\text{Hence, } 1 - \left(\frac{1}{r}\right)^{\gamma-1} > 1 - \left(\frac{1}{r}\right)^{\gamma-1}\left\{\frac{\rho^{\gamma} - 1}{\gamma(\rho - 1)}\right\}.$$

This result would suggest that a higher efficiency is attainable with an Otto engine than with a Diesel. But, in practice, a very much greater ratio of compression can be used in the Diesel than in the Otto engine, for the reason that in the Diesel engine a non-explosive substance—pure air—is compressed. In consequence of this the Diesel is somewhat more, not less, efficient than the Otto engine. On the other hand, its mechanical efficiency is less, so that, on the whole, there is not a great deal of difference in the net thermal efficiencies of these engines. It must always be borne in mind that the theoretical cycles are merely the best approximations to the actual "cycles", and that these latter are limited by many practical considerations which are ignored (because they do not arise) in the theoretical treatment. As already remarked, the internal combustion engine "cycles" are not true cycles (thermodynamically). The practical considerations are, of course, of first importance, as the cycles have to be actually carried out with engineering materials and appliances. Though one cycle be more efficient than another, theoretically, it by no means follows that the practical cycle corresponding to it will be more efficient than the other. It may be less efficient because of practical limitations.

Thus far, we have considered gases and have made some progress in discussing the thermodynamics of gases, using only:

1. The characteristic equations of perfect gases.
2. The conservation of energy and first law of thermodynamics.
3. The idea of entropy as the integral, along a reversible path, of heat taken in to a body divided by the temperature *of the body*.

When we proceed to deal with vapours we meet a new difficulty. Vapours can pass from the gaseous to the liquid, and even to the solid, phase and vice versa. This new difficulty must be overcome by a new principle. This new principle is *The Second Law of Thermodynamics*.

Before considering the second law, it may be well to make the following remarks:—

1. Entropy is a measurable property of the body itself that we are considering as the working substance. It depends on the temperature *of the body* and on its volume (see p. 340, equation 14).

2. On the other hand, the *Second Law of Thermodynamics* is a theorem about the absorption of heat energy from and delivery of heat energy to *other* bodies by the working substance. It involves the temperatures of these *other* bodies. These temperatures we shall denote by T's, whereas the temperatures of the working substance will be denoted by θ's.

3. When heat exchanges are *reversible*, the temperature of the working substance absorbing heat (say) is identical with that of the body supplying the heat—i.e. if T_1 is the temperature of the body supplying the heat and θ_1 of the working substance,

$$T_1 = \theta_1.$$

4. If the heat exchanges are *irreversible*, T_1 is not equal to θ_1.

The Second Law of Thermodynamics.

The second law of thermodynamics may be put in this way.

It is impossible, by any cyclic operation on a working substance, to transfer heat from a colder to a hotter external body unless energy is absorbed from external sources.

This conclusion is drawn from experience and must be accepted as an experimental fact. A further corollary follows from this fact, namely,

With given limits of temperature, no cycle is more efficient than a reversible cycle, no matter what working substance is used.

To prove this corollary we may proceed as follows:

Let C and C′ be ideal cylinders (fig. 7), with frictionless pistons P and P′. Let a hot body be

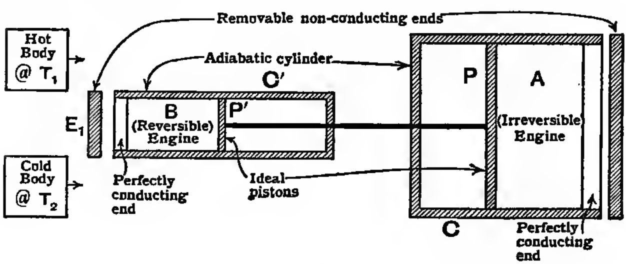


Fig. 7

available at T_1 and a cold body at T_2. Let B and A be volumes of *any* working substances (a gas, for example, though the working substance need not necessarily be a gas).

Assume it is possible, by means of the hot and cold bodies, to cause A and B to go through a

VOLTA (1745-1827)
From the portrait by Nicolo Bettoni

CARNOT (1796-1832)
From an engraving

HENRY (1797-1878)
From a steel engraving

CLAUSIUS (1822-1888)
From a photograph

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cycle which is reversible as regards B. How this can be done practically will be seen later.

Let the cycle to which A is subjected be reversible or not, as the case may be.

Let B, when working directly, absorb Q_1 heat units from the hot body and reject Q_2 heat units to the cold body. Then the amount of heat which disappears is $Q_1 - Q_2$, and this quantity, in heat units, must be the equivalent of the mechanical work done by the engine B, i.e. work is $(Q_1 - Q_2)$, and the efficiency of B is $\left(\frac{Q_1 - Q_2}{Q_1}\right)$.

Let A, when working directly, absorb H_1 heat units from the hot body and reject H_2 heat units to the cold body. The work done is $(H_1 - H_2)$ and the efficiency of A is $\left(\frac{H_1 - H_2}{H_1}\right)$.

Now, let the engine A drive engine B backwards, so that the net mechanical work done by A per cycle must equal the net mechanical work done *on* B per cycle. Since B is reversible, the result of reversing the B cycle is that Q_1 units of heat are rejected from the working substance to the hot body, and the working substance *absorbs* Q_2 heat units from the cold body.

Now, $Q_1 - Q_2 = H_1 - H_2$ (= W, say) since the mechanical work done *by* A is equal to that done *on* B.

Also, if A has a higher efficiency than B,

$$\frac{H_1 - H_2}{H_1} > \frac{Q_1 - Q_2}{Q_1},$$

$$\text{i.e. } \frac{W}{H_1} > \frac{W}{Q_1}.$$

$$\therefore H_1 < Q_1,$$

$$\text{and } H_2 < Q_2 \text{ (since } Q_1 - Q_2 = H_1 - H_2).$$

No mechanical energy from an external source is supplied to this apparatus, yet it transfers $(Q_1 - H_1)$ units of heat *to* the hot body, and absorbs $(Q_2 - H_2)$ units of heat from the cold body, *and each of these quantities is positive*, since

$$Q_1 > H_1 \text{ and } Q_2 > H_2.$$

Therefore *the hot body is getting hotter and the cold body colder, and during the process no energy is being supplied from an external source.*

This is contrary to the Second Law of Thermodynamics. Therefore the efficiency of A is not greater than the efficiency of B, and η_A must be either equal to or less than η_B, i.e.

$$\eta_A \leqq \eta_B.$$

The result obtained is *quite independent of the working substance employed.*

If A is reversible, then, by using B to drive A backwards, we can show by the same reasoning as before that if A is less efficient than B, heat will be transferred from the cold body to the hot body *without any work being done on the system from outside.* It follows that A is not less efficient than B.

Therefore all reversible cycles working between the same temperatures T_1 and T_2 have equal efficiencies. *This efficiency is the maximum efficiency possible between the given temperature limits, and is independent of the properties of the working substance.*

To find what this efficiency is, it is only necessary to study in detail the simplest case.

Carnot's Cycle.

The simplest reversible cycle, using *one* upper temperature and *one* lower temperature, is Carnot's. (The theoretical Otto and Diesel cycles require a series of hot and cold bodies at progressive temperatures.)

The simplest working substance is a gas.

Suppose a volume of gas is initially in the state specified by p_a, v_a, and θ_a, where $\theta_a = T_1$, and is contained in a cylinder whose sides are impervious to heat, and whose end is a perfect conductor of heat. The piston is supposed to be impervious to heat and to be frictionless (fig. 8).

1. Let the gas expand isothermally at θ_a from A to B by applying the hot body T_1 to the end of the cylinder, and very gradually reducing the balancing force F.

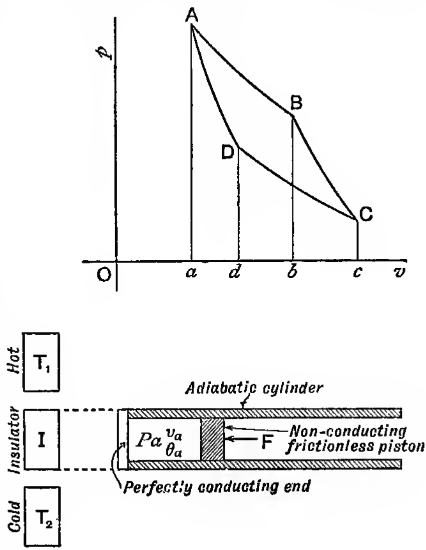

Fig. 8

The gas must expand to prevent the temperature from rising, and must therefore do mechanical work.

There is no change in the internal energy E, and the mechanical work done by the gas is measured by the area AB*ba*.

Therefore, by the conservation of energy, heat must have been supplied.

Suppose Q_1 units of heat were supplied at T_1, the temperature of the hot body.

2. Let the gas expand adiabatically from B to C, by placing I, a heat insulator (fig. 8) against the back. and very gradually reducing the balancing force F.

No heat is supplied.

The mechanical work of expansion done by the gas is measured by the area BC*cb*. Hence the internal energy must have dropped, i.e. the temperature must have fallen. Suppose the expansion is such that the temperature falls to T_2, the temperature of the cold body.

3. Let the gas be compressed isothermally from C to D by increasing the external force slowly, D being on the adiabatic through A. The cold body T_2 is applied to the end of the cylinder during this compression.

The work done on the gas is DC*cd*. There is no change in internal energy, hence the gas must have given out heat, say Q_2 units at T_2.

4. Let the gas be compressed adiabatically, with I (fig. 8) in contact with the end of the cylinder, from D to A by increasing the external force gradually.

No heat exchange takes place, but mechanical work is done on the gas equal to AD*da*. Its internal energy must have increased, i.e. its temperature must have risen. Since the adiabatic through D passes through A, the gas will, after a certain amount of adiabatic compression along DA, reach its initial condition A, and we have a cycle of operations.

The θ, ϕ diagram is simply a rectangle, the width of which is $R' \log_e\left(\frac{v_b}{v_a}\right)$ and the height $(\theta_a - \theta_d)$.

The net (reversible) mechanical work done by the gas is represented by the area ABCD.

The internal energy of the gas has not changed because the gas has gone through a cycle, and hence this work could only have been obtained at the expense of the heat energy of which the gas is the vehicle. This heat energy is

$$(Q_1 - Q_2) = \text{area ABCD}$$

(in consistent units), and this amount of heat has been transformed into mechanical work.

The *efficiency* of the cycle is $\dfrac{Q_1 - Q_2}{Q_1}$.

We can calculate this quantity in the following way: All the heat exchanges are reversible, hence

$T_1 = \theta_a$, the temperature of the gas along AB,
and $T_2 = \theta_c$,, ,, ,, CD.

The mechanical work done, A to B,

$$= R'T_1 \log_e\left(\frac{v_b}{v_a}\right) \text{ (p. 326).}$$

$$\therefore Q_1 = R'T_1 \log_e\left(\frac{v_b}{v_a}\right).$$

$$\text{Similarly, } Q_2 = R'T_2 \log_e\left(\frac{v_c}{v_d}\right).$$

But since AD and BC are portions of adiabatic lines,

$$T_1 v_a^{\gamma-1} = T_2 v_d^{\gamma-1},$$

$$\text{and } T_1 v_b^{\gamma-1} = T_2 v_c^{\gamma-1}, \text{ by equation (11).}$$

$$\therefore \frac{v_b}{v_a} = \frac{v_c}{v_d} = r.$$

$$\therefore \frac{Q_1 - Q_2}{Q_1} = \frac{R'(T_1 - T_2)\log_e r}{R'T_1 \log_e r}.$$

$$\therefore \frac{Q_1 - Q_2}{Q_1} = \frac{T_1 - T_2}{T_1}.$$

Hence the efficiency (η) is given by

$$\eta = \frac{T_1 - T_2}{T_1}.$$

This is the fact to which the Second Law leads by considering a Carnot cycle, namely, that no cycle working between temperatures T_1 (constant) and T_2 (constant) can have a greater efficiency than

$$\frac{T_1 - T_2}{T_1}.$$

Since $\frac{Q_1 - Q_2}{Q_1} = \frac{T_1 - T_2}{T_1}$, it follows that $\frac{Q_1}{T_1} = \frac{Q_2}{T_2}$ numerically.

If we take heat *entering* the gas as positive, Q_2 is negative. Hence

$$\frac{Q_1}{T_1} + \left(\frac{-Q_2}{T_2}\right) = 0,$$

$$\text{i.e } \Sigma\frac{Q}{T} = 0 \quad \ldots\ldots\ldots\ldots\ldots(19)$$

round the cycle.

In this result it should be carefully noted that T is the *absolute* temperature *of the external body* to which, or from which, heat passes. In a *reversible* cycle this temperature is the same as the temperature of the working substance when heat is being exchanged.

We have shown that the properties of the working substance do not affect the efficiency of the cycle. The efficiency depends simply on temperature—it is the same for the same temperatures, whether one or another substance is used as the working substance.

What part, then, does the working substance play? It must play some part, otherwise we could do without it altogether. It plays the part of a carrier of energy.

Carnot's theorem comes to this:

If mechanical work is done by subjecting an "energy-carrier" to a cycle of changes in which the initial and final states are the same, and in which heat is absorbed at one constant temperature from a hot body, and rejected at a lower constant temperature to a cold body, then the maximum efficiency of energy conversion is

$$\frac{T_1 - T_2}{T_1}; \text{ and } \Sigma\frac{Q}{T} = 0;$$

where T_1 is the temperature of the hot body, T_2 the temperature of the cold body, and Q is heat *entering* the energy-carrier.

This idea opens the road to the *Thermodynamics of Radiation*.

In 1875 Bartoli pointed out that space containing radiation is an energy-carrier, and therefore that such space can be treated as the "working substance" of a heat engine. The idea has been a most fruitful one in the hands of Rayleigh, Jeans, Boltzmann, Wien, and Planck, and is the starting point in the modern theory of radiation.

Value of $\Sigma\frac{Q}{T}$ for any Reversible Cycle.

The Second Law requires the efficiency of all reversible cycles, working between given limits of temperature, to be equal. Hence, for *any* reversible cycle, using *any* working substance,

$$\frac{Q_1 - Q_2}{Q_1} = \text{efficiency of Carnot's cycle}$$

$$= \frac{T_1 - T_2}{T_1}.$$

$$\therefore \frac{Q_1}{T_1} = \frac{Q_2}{T_2},$$

$$\text{and } \Sigma\frac{Q}{T} = 0,$$

taking heat *into* the working substance as positive.

This result is true for any reversible *simple* cycle, i.e. for a cycle using one upper temperature and one lower temperature only.

Generalization of Carnot's Cycle.

Suppose a working substance undergoes any reversible cyclic change, such as AB (fig. 9).

Such a cycle can only be reversible if we suppose we have at our disposal sources and sinks of heat at every temperature between the maximum and minimum temperatures of the working substance.*

We may then suppose that every exchange of heat is between the working substance and a body at the same temperature, i.e. $T = \theta$ at every point of the cycle.

We can draw over the diagram in fig. 9 a system of isothermal and adiabatic lines, a few of which are shown in the figure.

* Inasmuch as the conditions of reversibility are external conditions, any series of changes of temperature, pressure, and volume in a body can be imagined to be carried out reversibly by suitably adjusting the external surroundings.

—Poynting and Thomson's *Heat*, Fifth Edition, p. 274.

Any one of the strips, such as *abcd*, is bounded by isothermals and adiabatics, and hence the elementary cycle, *abcd*, is a Carnot cycle, and if $(\delta Q)_1$ is

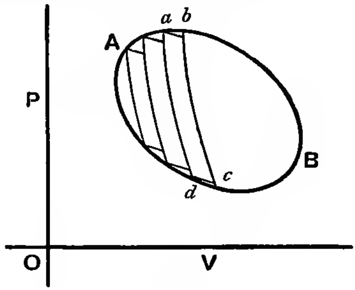

Fig. 9

the heat absorbed from the external body at T_1 and $(\delta Q)_2$ is the heat passing from the working substance to the external body at T_2, then

$$\frac{(\delta Q)_1 - (\delta Q)_2}{(\delta Q)_1} = \frac{T_1 - T_2}{T_1}$$

by Carnot's Theorem, i.e.

$$\frac{(\delta Q)_1}{T_1} = \frac{(\delta Q)_2}{T_2},$$

$$\text{i.e. } \frac{(\delta Q)_1}{\theta_1} = \frac{(\delta Q)_2}{\theta_2},$$

since $T = \theta$ everywhere because the cycle is reversible.

(This is the point in the argument where *the temperature of the working substance* is introduced instead of the temperature of the external body.) We have therefore

$$\frac{(\delta Q)_1}{\theta_1} - \frac{(\delta Q)_2}{\theta_2} = 0.$$

But, if we take heat passing *into* the working substance as positive, $(\delta Q)_2$ must be written $-(\delta Q)_2$, and then

$$\frac{(\delta Q)_1}{\theta_1} + \frac{(\delta Q)_2}{\theta_2} = 0.$$

A similar equation holds for the neighbouring Carnot strip. Two neighbouring strips combined are equivalent to a cycle represented by the boundary line of the combined strip, because the common boundary between the strips is traversed in one direction when considered as belonging to one strip and in the other direction for the other strip. In the same way the cycle represented by the zigzag boundary line of all the strips may be considered as the sum of all the elementary Carnot cycles into which the cycle is divided, and for these cycles we have

$$\Sigma\frac{\delta Q}{\theta} = 0,$$

where Σ stands for a summation for all the Carnot cycles.

By making $\delta Q \rightarrow 0$, i.e. by making the width of the elementary Carnot cycles $\rightarrow 0$, we can make the diagram of the compound Carnot cycle differ as little as we please from that of the given cycle, and hence for the given cycle

$$(\int)\frac{dQ}{\theta} = 0. \quad \text{.....................(20)}$$

This result is the simplest mathematical statement of the Second Law of Thermodynamics.

In any Reversible Cycle.

$$(\int)\frac{dQ}{T} = (\int)\frac{dQ}{\theta}. \quad \text{.....................(21)}$$

Consequences of $(\int)\frac{dQ}{\theta} = 0$ for a Reversible Cycle.

Let A and B (fig. 10) represent two states of the working substance, which can be reached by different reversible paths I and II.

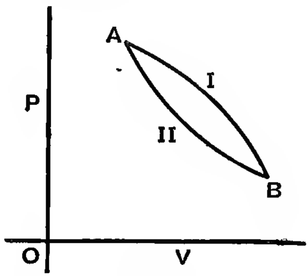

Fig. 10

Then AIBII is a reversible cycle, and

$$(\int)\frac{dQ}{\theta} = 0.$$

$$\therefore \int_A^B \left(\frac{dQ}{\theta}\right)_I + \int_B^A \left(\frac{dQ}{\theta}\right)_{II} = 0,$$

where the suffixes indicate integration along the paths I and II respectively.

$$\therefore \int_A^B \left(\frac{dQ}{\theta}\right)_I = \int_A^B \left(\frac{dQ}{\theta}\right)_{II},$$

i.e. $\int_A^B \frac{dQ}{\theta}$ is the same for any *reversible* path along which it may be integrated. Now, as there are innumerable reversible paths by which a substance in state A can pass into state B, $\int_A^B \frac{dQ}{\theta}$ can depend only on these states, so that, provided the initial and final states A and B are the same, the integral has the same definite value, whatever reversible path is actually followed.

It follows that $\int_A^B \frac{dQ}{\theta}$ is a function of the numbers which define the states A and B.

We may therefore write $\int_A^B \frac{dQ}{\theta} = f(A, B)$.

We can now show that the function f has a particularly simple form. Take a standard state O. Then

$$\int_0^B \frac{dQ}{\theta} = \int_0^A \frac{dQ}{\theta} + \int_A^B \frac{dQ}{\theta},$$

for a reversible path from O to B can be taken through A.

$$\text{Hence } \int_A^B \frac{dQ}{\theta} = \int_0^B \frac{dQ}{\theta} - \int_0^A \frac{dQ}{\theta}$$

= (a function of B) − (the same function of A)

$= \phi_B - \phi_A$, say,

where $\phi_X = \int_0^X \frac{dQ}{\theta}$.

Consequently,

$$\phi_B - \phi_A = \int_A^B \left(\frac{dQ}{T}\right)_r = \int_A^B \frac{dQ}{\theta},$$

where the suffix r denotes a reversible path of integration.

The function of state, ϕ, is called the entropy.

We are directly led to it by the Second Law of Thermodynamics applied to reversible cycles.

It is here proved to exist *for any working substance*, and to be a definite function of the state of the working substance. In fact, for steam, CO_2, SO_2, &c.,

tables of ϕ exist, so that we can look up the value of ϕ for any one of these substances in any specified state.

Definition of Entropy.

Since any series of changes of pressure, volume, and temperature in a body can be imagined to be carried out reversibly by suitably adjusting the external surroundings (p. 360), the increase in entropy between state A and state B is $\int \frac{dQ}{\theta}$ taken along *any* reversible path from state A to state B.

An Irreversible Cycle.

If $\int_A^B \left(\frac{dQ}{T}\right)_i$ be taken along an irreversible path it no longer necessarily measures $(\phi_B - \phi_A)$.

The efficiency of an *irreversible* cycle cannot be greater than the efficiency of Carnot's cycle, assuming that each cycle works between T_1 and T_2. Carnot proved this (p. 352).

$$\therefore \left(\frac{Q_1 - Q_2}{Q_1}\right)_i \overline{<} \frac{T_1 - T_2}{T_1},$$

where the suffix i refers to an irreversible cycle.

$$\therefore \frac{Q_2}{Q_1} \overline{>} \frac{T_2}{T_1}.$$

$$\therefore \frac{Q_2}{T_2} \overline{>} \frac{Q_1}{T_1}.$$

$$\therefore 0 \overline{>} \frac{Q_1}{T_1} - \frac{Q_2}{T_2},$$

or, regarding heat put *into* the working substance as positive,

$$0 \overset{=}{>} \Sigma\frac{Q_1}{T_1}$$

for a simple cycle, and

$$\left(\int\right)\frac{dQ}{T} \leqq 0 \text{ ....(22)}$$

for any cycle (cp. p. 362).

We use the "equals" sign for a reversible cycle and the *combined* sign for an irreversible cycle.

An Irreversible Path.

If the states A and B are connected by a reversible path r and an irreversible path i (fig. 11), we have

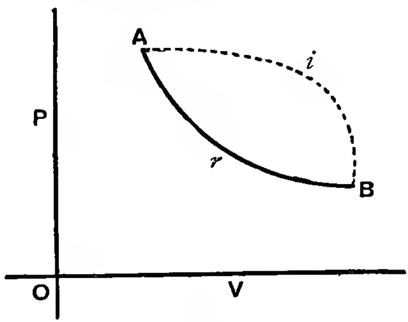

Fig. 11

$$\left(\int\right)\frac{dQ}{T} \overset{=}{<} 0.$$

$$\therefore \int_A^B\left(\frac{dQ}{T}\right)_i + \int_B^A\left(\frac{dQ}{T}\right)_r \overset{=}{<} 0.$$

$$\therefore \int_A^B\left(\frac{dQ}{T}\right)_i - \int_A^B\left(\frac{dQ}{T}\right)_r \overset{=}{<} 0.$$

$$\therefore \int_A^B \left(\frac{dQ}{T}\right)_i \overline{\overline{<}} \int_A^B \left(\frac{dQ}{T}\right)_r.$$

$$\therefore \int_A^B \left(\frac{dQ}{T}\right)_i \overline{\overline{<}} \phi_B - \phi_A,$$

i.e. $\int_A^B \frac{dQ}{T}$, for an irreversible change of state, cannot be greater and may be less than the change in entropy corresponding to the given change of state.

If the state B is indefinitely near to A, we can write

$$\delta Q \overline{\overline{<}} T\delta\phi$$

for an irreversible infinitesimal change of state.

An Adiabatic Reversible Change of State.

Suppose the working substance passes from state A to state B reversibly and adiabatically, then, since the path is reversible, by hypothesis, $\int_A^B \frac{dQ}{\theta}$ measures the increase in entropy from state A to state B, if taken *along the actual path.* But, along the actual path, δQ is everywhere zero, because the change of state takes place adiabatically.

$$\therefore \int_A^B \frac{dQ}{\theta} = 0,$$

i.e. there is no change of entropy.

An adiabatic reversible change of state is therefore an isentropic change of state.

It should be carefully noted that nothing has been said as to whether the change of state takes place by expansion or compression.

The conclusion reached is therefore equally true whether the change of state is brought about by expansion or compression.

An Irreversible Adiabatic Change of State.

Now, suppose the change of state from state A to state B is *irreversible* and adiabatic, then

$$\int_A^B \frac{dQ}{\theta} = 0$$

when taken along the actual path, because δQ is everywhere zero, by hypothesis. But this integral does *not* measure the change in entropy when taken along the actual path, because the actual path is irreversible.

As proved on p. 367,

$$\int_A^B \frac{dQ}{T} \overline{\overline{<}} \phi_B - \phi_A,$$

where the integral is taken along the actual path, and this integral is zero just as $\int_A^B \frac{dQ}{\theta}$ is zero, *along the actual path.*

$$\therefore \phi_B - \phi_A \geqq 0,$$

i.e. there can be no decrease and may be an increase of entropy.

An adiabatic irreversible change of state is therefore not necessarily isentropic but may be. The entropy cannot decrease—that is all we can say.

Just as in the previous paragraph, this conclusion is equally true whether the change is brought about by expansion or compression.

PRACTICAL EXAMPLES

1. Suppose a given volume of a perfect gas at p, v, θ expands adiabatically into a vacuum so that its final state is p_1, v_1, and θ_1. Then, it is an experimental fact (Joule and Thomson's experiments, p. 338) that θ_1 is very nearly equal to θ.

The expansion is clearly irreversible, and since $\theta = \theta_1$ it is isothermal, and it is evident from the p, v and θ, ϕ diagrams that we can change from state A to state B by a reversible isothermal expansion in which the entropy increases.

In fact, the increase of entropy, per pound, is

$$C_v\int_\theta^{\theta_1}\frac{d\theta}{\theta}+R'\int_v^{v_1}\frac{dv}{v},$$

i.e. $0 + R' \log_e\frac{v_1}{v}$, which is positive.

2. When steam expands in a turbine, the expansion is irreversible and nearly adiabatic. The entropy of the steam, per pound, when it leaves the turbine is greater than when it enters it. This is an experimental fact.

3. When air is compressed in an air compressor, the compression of the air is irreversible and nearly adiabatic. The entropy, per pound, of the air when it leaves the compressor is greater than when it enters the compressor. This, again, is an experimental engineering fact.

We will now consider changes of state when no mechanical work is done.

A Change of State brought about by a Reversible Communication of Heat.

The heat added to the working substance is δQ say. It is added *reversibly* at θ, the temperature of the working substance, hence $\frac{\delta Q}{\theta}$ is the increase of entropy, i.e. for the complete change

$$\phi_B - \phi_A = \int_A^B \frac{dQ}{\theta},$$

where the integral is carried out with reference to the *actual* heat exchange.

A Change of State brought about by an Irreversible Communication of Heat.

We cannot now integrate along the actual heat exchange to find the change of entropy.

The substance changes in the irreversible process from state A (p_0, v_0, θ_0) to state B (p_1, v_1, θ_1), and to find the increase in entropy, we must integrate $\int_A^B \frac{dQ}{\theta}$ *along any reversible path* which will bring the substance from state A to state B.

This integral is not *necessarily* equal to $\int_A^B \frac{dQ}{\theta}$ *for the actual path* as the increments of heat might be different.

Changes in the Entropy of a System.

So far, we have confined ourselves to one body, the working substance. But every other body of the system may undergo changes of state, and has a definite entropy which may or may not change.

Suppose two bodies a and b interact in any way, so that the state of a changes from A to A′ and of b from B to B′.

1. Suppose it is merely an exchange of heat, then it may be reversible or irreversible.

If it is reversible θ_a, the temperature of a, equals θ_b, the temperature of b.

The gain of entropy of A, due to the passage of δQ of heat from A to B, is $-\frac{\delta Q}{\theta_a}$, and the gain of entropy of B is $+\frac{\delta Q}{\theta_b}$.

As $\theta_a = \theta_b$, the net gain of entropy is nil.

If the exchange of heat is irreversible, the gain in entropy by one body may exceed the loss of entropy by the other, and, consequently, the exchange of heat under irreversible conditions between the bodies of the system may increase the total entropy of the system. As an instance, we will take the irreversible exchange of heat by conduction which takes place when 1 lb. of water is heated from a temperature θ_0 to θ_1 by means of a hot body of very great mass at $T_1 = \theta_1$.

The heat exchanged is $c(\theta_1 - \theta_0)$ heat units, where c is the specific heat of water, viz. unity.

The hot body therefore loses $(\theta_1 - \theta_0)$ heat units at constant temperature T_1.

Hence it loses entropy $\left(\frac{\theta_1 - \theta_0}{T_1}\right)$ units by the equivalent reversible change.

On the other hand, the water gains entropy equal to

$$\int_{\theta_0}^{\theta_1} \frac{d\theta}{\theta} = \log_e \frac{\theta_1}{\theta_0},$$

by the equivalent reversible change, and

$$\frac{\theta_1 - \theta_0}{T_1} = \frac{\theta_1 - \theta_0}{\theta_1}$$

since $T_1 = \theta_1$. But

$$\frac{\theta_1 - \theta_0}{\theta_1} < \log_e \frac{\theta_1}{\theta_0} \text{ (see below)},$$

hence the gain of entropy by the water is greater than the loss of entropy by the hot body, and the *algebraic* gain of entropy is

$$\left[\log \frac{\theta_1}{\theta_0} - \frac{\theta_1 - \theta_0}{\theta_1}\right]$$

for the system consisting of the two bodies.

This quantity is positive, for $\int_{\theta_0}^{\theta_1} \frac{d\theta}{\theta}$ is greater than it would be if $\frac{1}{\theta}$, under the integral sign, were replaced by its *least* value $\frac{1}{\theta_1}$. (See p. 72.) Hence we conclude that

by an irreversible exchange of heat we may increase the total entropy of the system consisting of the body being heated and the body heating it.

2. Suppose the exchange is one of mechanical energy without exchange of heat, i.e. it is an adiabatic change. Then one of the bodies expands doing work on the other which contracts. In either case, by p. 368, the entropy of each body must either remain unchanged or increase.

We have seen that in the two examples considered, viz. an exchange of heat without mechanical work being done and an exchange of mechanical work without heat exchange, the entropy of the system cannot decrease. It may increase if the changes are irreversible; otherwise it is unaltered. These conclusions have been found to hold good for all cases which have been examined in detail by physicists and chemists, and have led to a principle of physics. This principle, which is generally accepted, is as follows:—

In all natural changes of state which may take place in the bodies of an isolated finite system, the total entropy of the system never decreases and generally increases.

Suppose that the entropy of a system of bodies is a maximum. This means that any hypothetical slight change of state in the system diminishes the entropy; but we have just seen that, in natural spontaneous changes, the entropy can never diminish. Hence the hypothetical slight change cannot occur. The system is therefore in stable equilibrium.

The tendency of a natural system of bodies is therefore always towards a state of greater entropy.

The *Principle of Maximum Entropy*, as it is called, is the starting point of much of the work on the applications of thermodynamics to chemistry.

The Steam-engine Cycle.

We shall suppose the engine to be a "condensing" engine, i.e. one in which the steam, after passing through the engine, is condensed and returned to the boiler. The working substance—sometimes steam, sometimes water—goes through the following cycle of operations. After leaving the condenser, the water is pumped by the boiler-feed pump into the boiler. It is there heated and turned into steam. It then passes through the engine and enters the condenser. It is there condensed by means of cold water, which removes the latent heat of the low-pressure steam.

In the boiler furnace, the heat is generated by combustion at a very high temperature, say 2000° F., whereas the highest temperature attained by the steam is comparatively small, say 800° F., even when a "superheater" is used. The transference of heat from the "hot body"—the furnace gases—to the working substance is therefore *irreversible.* The lowest temperature of the working substance depends upon the vacuum in the condenser. If the condenser pressure is 1 in. Hg. say, the temperature is about 80° F., the saturation temperature at that pressure. The lowest temperature of the cooling water—the "cold" body—is about 15° F. lower than the lowest temperature of the working substance, hence the passage of heat from the working substance to the cold body is *irreversible.* The external work is also done by the working substance irreversibly. The actual cycle is therefore essentially an irreversible one. We can, however, imagine a *reversible* cycle which would correspond to the actual cycle, if all the exchanges of heat and mechanical energy could be carried out reversibly. This ideal reversible cycle was proposed by Rankine and is named after him. It is a compound Carnot cycle.

Before describing this cycle we must deal with a point of difference between the steam cycle and any cycle we have hitherto considered.

Working Substance not all in the Same State Simultaneously.

The working substance is not all in the same state at any given moment; for instance, the steam in the boiler is in a very different state from the water in the condenser. We must therefore consider an element of the working substance of mass δm, and follow its history as it passes round the cycle outlined above.

Rankine's Cycle.

The steam when it leaves the boiler may be wet, dry, or, if the boiler is fitted with a superheater, superheated, i.e. at a temperature higher than the saturation temperature corresponding to the boiler pressure. These three possibilities lead to three different sets of equations, but for simplicity we shall take the single case when the steam is initially dry when it leaves the boiler. The principles upon which the calculations are based are exactly the same whether there is initial wetness or initial superheat or neither. Suppose fig. 12 is a p, v diagram representing the state of a given mass, 1 lb. say, at different points of the ideal cycle.

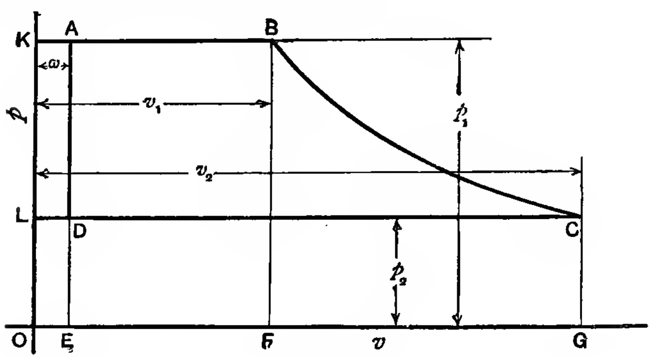

Fig. 12

The point A represents the state of the substance in the boiler before its latent heat is imparted to it. We shall denote this state by p_1, ω, θ_1.

The point B represents the state after evaporation (p_1, v_1, θ_1).

The point C represents the state after reversible adiabatic expansion in the engine (p_2, v_2, θ_2).

The point D represents the state after condensation to water in the condenser (p_2, ω, θ_2).

We can now suppose that the mass, 1 lb., goes through this cyclic change of state in a cylinder, i.e. in a "one-organ" engine.

The actual engine has several organs, namely, boiler, condenser, &c; indeed, the idea of using several organs and keeping one organ for one purpose was Watt's great contribution to the practical development of the steam-engine. In the *ideal* theory, it makes no difference in principle whether we suppose the engine to have one or several organs, but the calculation is easier to follow if we suppose the cyclic change to take place in a cylinder.

The steps in the cycle are as follows:

1. The water (at state D) is heated up from θ_2 to θ_1, the pressure on the piston being increased so as to correspond to the saturation pressure of the steam at each temperature.

2. Reversible evaporation under constant pressure p_1 by a hot body at temperature θ_1.

3. Reversible adiabatic expansion from pressure p_1 to pressure p_2, with consequent cooling from θ_1 to θ_2, and some condensation.

4. Reversible complete condensation under constant pressure p_2 by a cold body at temperature θ_2.

The figure ABCD (fig. 12) is the indicator diagram for this cycle, and represents the work done per pound per cycle.

The work done by the element δm is then

$$\delta m \times \text{area ABCD} \times f,$$

where f is a scale factor.

The area ABCD can be obtained by integration, but it is much easier to obtain it from the theorem that $(\int)d\phi = 0$. Steps 2, 3, and 4 above are reversible. A reversible step corresponding to step 1 is obtained if we suppose the feed-water heating is done by means of a series of progressively hot external bodies, each of which is at the temperature of the water it is heating. The increase of entropy along this path is then $\int \frac{dQ}{T}$, where the integral is taken from the state corresponding to D to that corresponding to A (fig. 12).

But $\delta Q = \delta\theta$ for water and $T = \theta$ as the step is reversible,

$$\therefore \int_D^A \frac{dQ}{T} = \int_{\theta_2}^{\theta_1} \frac{d\theta}{\theta} = \log_e \frac{\theta_1}{\theta_2}.$$

We get, then, for the *increases* of entropy in the different steps:

1. $+ \log_e \frac{\theta_1}{\theta_2}$.
2. $+ L_1/\theta_1$.
3. None.
4. $- x_2 L_2/\theta_2$,

where x_2 is the dryness fraction of the steam at state C, hence

$$\frac{x_2 L_2}{\theta_2} = \frac{L_1}{\theta_1} + \log_e \frac{\theta_1}{\theta_2}. \quad \text{.............(23)}$$

The entropy diagram is shown in fig. 13.

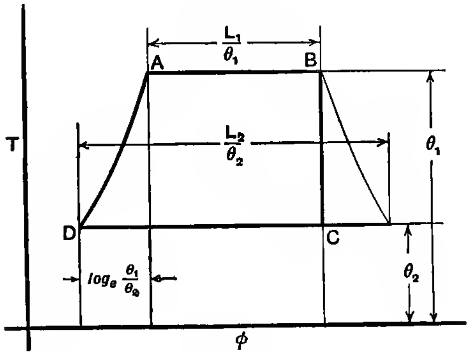

Fig. 13

Equation (23) gives us x_2 as all the other quantities are known, i.e.

$$x_2 = \frac{\theta_2}{L_2}\left[\frac{L_1}{\theta_1} + \log_e\frac{\theta_1}{\theta_2}\right] \ldots\ldots\ldots\ldots (24)$$

The heat taken in, per pound, by the working substance is

$$[(\theta_1 - \theta_2) + L_1] \text{ B.Th.U.}$$

The heat given up to the condenser is x_2L_2 B.Th.U., per pound, i.e.

$$\theta_2\left[\frac{L_1}{\theta_1} + \log_e\frac{\theta_1}{\theta_2}\right] \text{ B.Th.U.,}$$

per pound, on substituting the value of x_2 given in equation (24).

Hence the efficiency (η) of the cycle is given by

$$\eta = \left[\frac{(\theta_1 - \theta_2) + L_1 - \theta_2\left(\log_e\frac{\theta_1}{\theta_2} + \frac{L_1}{\theta_1}\right)}{(\theta_1 - \theta_2) + L_1}\right] \ldots (25)$$

The work done, per pound (W), by the working substance is the numerator of this fraction, i.e.

$$W = \left[(\theta_1 - \theta_2) + L_1 - \theta_2\left(\log_e \frac{\theta_1}{\theta_2} + \frac{L_1}{\theta_1}\right)\right] \text{B.Th.U.}$$

Suppose M is the "water-rate" of the engine, i.e. the number of pounds of steam flowing past the boiler stop valve *per second*. Then a mass M goes through the cycle per second. Now, a mass δm in going through the cycle does work, $W\delta m$ B.Th.U., therefore a mass M does (MW) B.Th.U. of external work, i.e. (MW) B.Th.U. of work are done *per second*, i.e.

$$\text{the H.P. developed} = \frac{MW \times 778}{550}$$
$$= 1{\cdot}415\ MW,$$

where W is the net external work, measured in B.Th.U.'s, done by a pound of steam in going through the cycle. The p, v and θ, ϕ diagrams, plotted for 1 lb., are therefore all we need to work out the efficiency and horse-power of the ideal Rankine cycle.

This cycle is the one with which the actual performances of steam-engines are compared. When the so-called "efficiency" of a steam-engine is given as 60 per cent say, it has 60 per cent of the efficiency of the Rankine cycle, the upper and lower temperatures of which are the same as those of the working substance in the *actual* engine.

Practical Calculation of the Rankine Cycle.

The steam engineer is constantly using the thermal values and efficiencies of Rankine cycles.

To simplify calculations, he uses a diagram of the properties of steam, plotted in terms of the "total heat" I, and the entropy ϕ.

This diagram is called the "Mollier" diagram for steam.

The "total heat" I is defined thus:

$$I = E + Apv, \qquad \text{.................(26)}$$

where E is the internal energy per pound, p the pressure, v the volume, and A is the constant 0·001285, i.e. $\frac{1}{778}$.

If the steam expands adiabatically from state B to state C, we have $\delta Q = \delta E + Ap\delta v$, by p. 332, equation (5) for any element, and from equation (26)

$$\begin{aligned} \delta I &= \delta(E + Apv) \\ &= \delta E + \delta(Apv) \\ &= \delta E + A\delta(pv) \\ &= \delta E + A[p\delta v + v\delta p], \text{ since} \\ \Delta(pv) &= (p + \Delta p)(v + \Delta v) - pv; \text{ hence} \\ \delta I &= \delta E + Ap\delta v + Av\delta p \\ &= 0 + Av\delta p, \end{aligned}$$

by the equation $\delta Q = \delta E + Ap\delta v$, since $\delta Q = 0$.

$$\therefore\ I_B - I_C = A\int_C^B v\,dp.$$

$$\text{Now, } \int_C^B v\,dp = \text{area KBCL (fig. 12),}$$

hence the area ABCD (fig. 12) is given by

$$\text{area ABCD} = \frac{I_B - I_C}{A} - (p_1 - p_2)\omega.$$

∴ the area ABCD in heat units is given by

$$(I_B - I_C) - A(p_1 - p_2)\omega.$$

In all but the most refined calculations, $A(p_1 - p_2)\omega$ is negligible compared to $I_B - I_C$, hence *the required area in heat units is simply the difference in the total heats at states B and C.* This quantity is called the "heat drop" between B and C. Tables of heat drops, for different initial and final conditions of temperature and pressure, have been calculated from Callendar's equation for steam (p. 326), by H. Moss (*Heat Drop Tables*, Arnold).

Exercise 16

1. Calculate the ideal efficiency of an Otto gas-engine, the compression ratio of which is 8.

2. Calculate the ideal efficiency of a Diesel cycle for which the compression ratio is 12, and $\rho = 3$.

3. Calculate the ideal efficiency of a Rankine cycle for a steam-engine working at 200 lb. per square inch abs. pressure, dry steam, and a condenser pressure of 1 in. Hg.

4. A pound of ice at 0° C. is put into a bucket of water at 10° C. The thermal capacity of the bucket and the water in it is equal to 20 lb. of water. Calculate the change in the entropy of the system due to the irreversible conversion of the ice into water. Take the pound calorie as the heat unit. The latent heat of liquefaction of ice is 80 C.H.U.

5. A system consists of a non-conducting cylinder, two gases, and a conducting piston between them. The surfaces of the piston in contact with the gases are non-conducting. The piston is a tight fit, so that the gas A, initially at a high pressure, expands very slowly against the gas B at the lower pressure. Show that the entropy of the system when equilibrium is reached must be greater than in the initial state of the system.

6. Show that the theoretical work, in foot-pounds, which must be done by an air-pump to take in, com-

press adiabatically, and expel 1 c. ft. of air against a pressure p_1 lb. per square inch, the initial pressure of air being p_2 lb. per square inch, is

$$144\frac{n}{n-1}p_2\left\{\left(\frac{p_1}{p_2}\right)^{\frac{n-1}{n}}-1\right\},$$

where pv^n = a constant is the equation of the adiabatic change of state.

7. Show that when the difference in pressure $(p_1 - p_2)$ in question 6 is small, the H.P. required, ideally, to compress V c. ft. per minute of air at pressure p_2 is

$$0 \cdot 00435\, V\delta p_2.$$

8. Show that, if the initial pressure in question 6 is atmospheric, the H.P. formula becomes

$$0 \cdot 2207\left\{\left(\frac{p_1}{p_2}\right)^{0 \cdot 29}-1\right\}V,$$

where V is the volume of atmospheric air dealt with in cubic feet per minute ($n = 1 \cdot 408$).

9. A gas-engine works on an ideal cycle, with adiabatic compression and expansion, receiving and rejecting heat at constant volume. The piston displacement per stroke is 1 c. ft., the clearance volume 0·2 c. ft., and at the beginning of compression the temperature of the cylinder contents is 600° F. absolute, the pressure being atmospheric. The engine receives 0·06 c. ft. of gas per cycle (calorific value 600 B.Th.U. per cubic foot). Atmospheric pressure = 14·7 lb. per square inch. Find—

(i) Weight of cylinder contents.
(ii) Pressure and temperature at end of compression ($\gamma = 1 \cdot 38$).
(iii) Rise of temperature during explosion (neglect jacket loss and take $C_v = 0 \cdot 18$).
(iv) Pressure at end of explosion.
(v) Temperature and pressure at end of expansion.

(vi) Efficiency of the cycle.

(vii) Efficiency of an engine working on a Carnot cycle between the same highest and lowest temperatures. (*Mech. Sc. Tripos*, 1906.)

10. The diagram is the indicator diagram taken on one side of the piston of a double-acting steam-engine. There was $\frac{1}{10}$ lb. steam present during expansion. Find, with

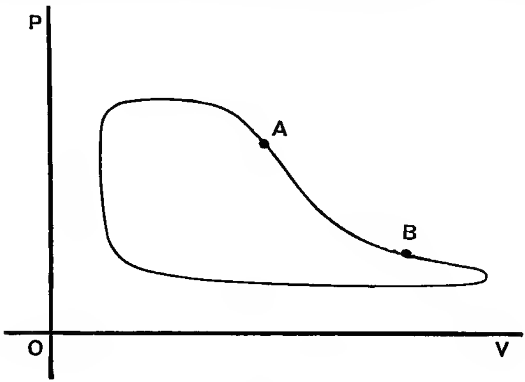

Fig. 14

the help of steam tables, its dryness fraction at the points A and B, also the change in internal energy and external work done between these points. Hence deduce the exchange of heat that must take place between the steam and the cylinder walls during the expansion from A to B.

The pressure and volume corresponding to the point A are 65 lb. per square inch absolute and 0·5 c. ft.

11. A boiler evaporates 30,000 lb. of water per hour at a coal consumption of 7 lb. of steam per pound of coal. The coal, in burning, uses 17·5 lb. of atmospheric air, per pound of coal. The products of combustion enter an electrically-driven chimney fan at 300° F. and 1 in. water-gauge of induced draught. The fan has an efficiency of 60 per cent. Calculate the output from the electric motor.

12. A torpedo air-chamber contains initially 80 lb. of air at a pressure of 1700 lb. per square inch absolute and 15° C., and at the end of the run the pressure is 500 lb. per square inch and the temperature 2° C. How much of the heat of the air which is left in the chamber has been abstracted from the sea? (*Mech. Sc. Tripos*, 1911.)

MISCELLANEOUS EXERCISES

1. Find the indefinite integrals of

$$\frac{1}{x(x-1)^2}; \quad x^3 \log x; \quad (x^2 - 3x + 2)^{-\frac{1}{2}};$$

and prove that

$$\int_0^\pi \frac{dx}{5 + 4\cos x} = \frac{\pi}{3}.$$

(*Int. B.Sc. Hons. Lond. 1904.*)

2. Find the following integrals:

$$\int \frac{dx}{e^x + e^{-x}}; \quad \int \sqrt{\frac{1-x}{1+x}}\, dx; \quad \int \frac{2x^2 - x + 10}{(x^2+4)(x-2)} dx.$$

(*B.Sc. Pass, Lond. 1903.*)

3. Differentiate with regard to x the function

$$\arctan \frac{x}{\sqrt{(1 - x^2)}}.$$

Effect the simplification of this function which the form of your answer suggests.

Of what function is $(1 - x^2)^{-\frac{1}{2}}$ the second derived function? (*B.Sc. Eng. Lond. 1906.*)

4. Find the following indefinite integrals:

$$\int \frac{dx}{x^3(x-1)}; \quad \int e^x \sin x dx; \quad \int x^3 e^x dx.$$

(*B.Sc. Eng. Lond. 1907.*)

5. Find the value of $\frac{dy}{dx}$ at any point x, y of the curve

$$x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}},$$

and write down the equation to the tangent to the curve at the point.

If the tangent cut the axes Ox, Oy in P and Q respectively, show that OP + OQ is constant and equal to a.

Take a equal to 4 in., and draw a number of straight lines cutting the axes and having the sum of their intercepts equal to this.

Show that the curve may then be sketched in to touch these lines. (*Qual. Exam. Mech. Sc. Trip. 1908.*)

6. Show by integration by parts that if I_n denotes the integral

$$\int_0^\beta e^{ax} \sin^n x dx,$$

where $\beta = \text{arc} \tan \frac{n}{a}$, we have

$$I_n = \frac{n(n-1)}{n^2 + a^2} I_{n-2}.$$

(*B.Sc. Pass, Lond. 1903.*)

7. Prove that if m and n be positive integers,

$$\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x dx = \frac{m-1}{m+n} \int_0^{\frac{\pi}{2}} \sin^{m-2} x \cos^n x dx,$$

and evaluate

$$\int_0^{\frac{\pi}{2}} \cos^4 x \sin^2 x dx \text{ and } \int_0^{\frac{\pi}{2}} \cos^5 x \sin^3 x dx.$$

(*B.Sc. Eng. Lond. 1907.*)

8. What properties of $f(x)$ may you infer from the graph of $f'(x)$? Amongst all right cones which have the same volumes, show that that one which has the least curved surface may be constructed from a circular sector whose angle contains 3·6 radians approximately.

(*B.Sc. Eng. Lond. 1906.*)

9. Prove that

$$\int_a^b (x - a)(x - b)[x - \tfrac{1}{2}(a + b)]^2 dx = \frac{(a - b)^5}{120}.$$

Apply Simpson's Rule to calculate an approximate value of $\log_e 10$ from the formula

$$\log_e 10 = \int_1^{10} \frac{dx}{x}.$$

(*B.Sc. Eng. Lond. 1906.*)

10. Integrate the functions

$$\frac{x^2 + 2x - 1}{x^2(x - 1)^2}, \quad \frac{1}{\sqrt{x(x - 2)}}, \quad \frac{1}{a^2 \sin^2 x + b^2 \cos^2 x}.$$

Show that, if $c > a > 0$,

$$\int_0^a \frac{\sqrt{a^2 - x^2}}{c^2 - x^2} dx = \pi(c - \sqrt{c^2 - a^2})/2c.$$

(*Maths. Trip. I. 1908.*)

11. Show that, when $f(x)$ is of the form $A + Bx + Cx^2$,

$$\int_0^1 f(x)dx = \tfrac{1}{6}\{f(0) + 4f(\tfrac{1}{2}) + f(1)\}.$$

Show also that, if $\phi(x)$ be any polynomial of the fifth degree,

$$\int_0^1 \phi(x)dx = \tfrac{1}{18}\{5\phi(\alpha) + 8\phi(\tfrac{1}{2}) + 5\phi(\beta)\},$$

(*Maths. Trip. I. 1909.*)

where α and β are the roots of $x^2 - x + \frac{1}{10} = 0$.

12. Find the differential coefficient (derivative) of

$$\text{(i) } \log_e \tan x; \quad \text{(ii) } \frac{e^{x^2+1}}{x^2 + 1}.$$

If $y = ax/(x + b)$ prove that

$$\frac{2}{y}\frac{dy}{dx} - \frac{d^2y}{dx^2}\Big/\frac{dy}{dx} = \frac{2}{x}.$$

(*Qual. Exam. Mech. Sc. Trip. 1910.*)

13. The co-ordinates of any point on a cycloid are given by the formulæ

$$x = a(\theta + \sin\theta),$$
$$y = a(1 - \cos\theta).$$

Prove that ϕ, the inclination of the tangent to the axis of x, is equal to $\frac{1}{2}\theta$, and that the length of the arc measured from the origin is given by $s = 4a \sin\phi$.

Prove also that the radius of curvature at any point is twice the intercept on the normal between the curve and the line $y = 2a$. (*B.Sc. Pass, Lond. 1911.*)

14. Evaluate the integral

$$\int \frac{x + a}{\{(x + \beta)(x + \gamma)\}^{\frac{1}{2}}} dx,$$

where a, β, and γ are real constants, and β is not equal to γ.

Prove that

$$\int_0^a \frac{dx}{(2a^2 - x^2)^{\frac{5}{2}}} = \frac{1}{3a^4}.$$

(*Int. B.Sc. Hons. Lond. 1913.*)

15. Apply the method of integration by parts to evaluate

$$\int \log_e x dx, \int x \log_e x dx, \int x e^{ax} dx.$$

Show that

$$\int_0^1 x \log_e(1 + \tfrac{1}{2}x) dx = \tfrac{3}{4}(1 - 2 \log_e \tfrac{3}{2}),$$

and prove in any way that this is less than

$$\int_0^1 \tfrac{1}{2} x^2 dx.$$

(*Maths. Trip. I. 1913.*)

16. Show how to find maximum and minimum values of a function of one variable.

Sketch the form of the curve $y = \dfrac{x}{1 + x + x^2}$, and determine the maximum and minimum values of y.

(*Qual. Exam. Mech. Sc. Trip. 1914.*)

17. State shortly rules for finding the maximum and minimum values of a function of x, and give reasons for the rules.

In the theory of transformers, the following expression for $\cos\psi$ occurs.

$$\cos\psi = \frac{(1 - \sigma)\sin 2\theta}{\{2(1 + \sigma^2) - 2(1 - \sigma^2)\cos 2\theta\}^{\frac{1}{2}}},$$

where $\sigma < 1$; prove that when $\cos\psi$ is a maximum

$$\cos 2\theta = \frac{1 - \sigma}{1 + \sigma} = \cos\psi.$$

(*B.Sc. Eng. Glasgow, 1917.*)

18. Evaluate the integrals

$$\int\frac{xdx}{(x + 1)^2(x - 1)}, \quad \int\cos^2 x dx.$$

$$\text{If } s_n = \int\frac{\sin(2n - 1)x}{\sin x}dx, \quad v_n = \int\frac{\sin^2 nx}{\sin^2 x}dx,$$

prove the reduction formulæ,

$$n(s_{n+1} - s_n) = \sin 2nx, \quad v_{n+1} - v_n = s_{n+1},$$

and show that if v_n is taken between the limits 0 and $\frac{1}{2}\pi$, its value is $\frac{1}{2}n\pi$, when n is an integer.

(*Maths. Trip. I. 1914.*)

19. Sketch the curve $2y^2 = x(1 - x)^2$, and find the volume of the solid obtained by rotating the loop about the axis of x. (*Qual. Exam. Mech. Sc. Trip. 1919.*)

20. Investigate carefully the conditions that $f(a)$ should be a minimum value of the function $f(x)$.

The total work done in a compound air compressor is given by the equation

$$W = K^2\left\{\left(\frac{p_1}{p}\right)^{\frac{n-1}{n}} + \left(\frac{p}{p_2}\right)^{\frac{n-1}{n}} - 2\right\},$$

where p, the pressure, is the variable; show that the work done is a minimum when $p = \sqrt{p_1 p_2}$.

(*B.Sc. Eng. Glasgow, 1915.*)

21. A spherical raindrop, whose radius is 0·04 in., begins to fall from a height of 6400 ft., and during the fall its radius grows by precipitation of moisture at the rate of 10^{-4} in. per second. Prove that, if its motion is unresisted, its radius on reaching the ground will be 0·04205 . . . in., and that it will have taken about 20·5 sec. to fall.

(*Int. B.Sc. Hons. Lond. 1904.*)

22. Trace the portion of the curve $y = 8 - x^3$ for which both x and y are positive. Without further calculation draw the tangent which is equally inclined to the axes. Measure the intercepts made by the tangent on the axes.

Find the equation of the tangent at the point whose coordinates are a, b. Calculate to two significant figures the values of a, b, for which the tangent is equally inclined to the axes, and the values of the intercepts it makes upon the axes.

Compare these values of the intercepts with those found from your drawing.

Find by integration the area enclosed by the curve and the axes, and check your result by counting squares.

(*Army Entrance, Higher Maths. 1915.*)

23. The displacement of the piston of an engine from the mid position is given by

$$y = r\cos\omega t - \frac{r^2}{4l}(1 - \cos 2\omega t),$$

where ω is the angular velocity of the crank,
r the crank radius,
and l the length of the connecting rod.

If M is the mass of the piston, find the maximum inertia force arising from it.

24. In the last example, state the percentage of the maximum inertia force which arises from the obliquity of

the connecting rod. Why is the effect of obliquity so much greater in disturbing the inertia force of the piston than it is in affecting its velocity?

25. A fly-wheel, of weight 1 ton and radius of gyration 3 ft. 6 in., is rotating once every second. What is its kinetic energy, and how long will it take to come to rest under a frictional torque round the axis of 40 lb.-ft.?

(*Qual. Exam. Mech. Sc. Trip. 1919.*)

26. The angular displacement of a pendulum is given by

$$\theta = \theta_0 e^{-kt} \sin nt.$$

Show that the successive maxima of θ form a series in geometrical progression.

If the time of a complete oscillation is 1 sec., and if the ratio of the first and the fifth angular displacements on the same side is 4:1, show that the time taken in swinging out from the position of equilibrium to an extreme displacement is 0·241 sec. (*Maths. Trip. I. 1917.*)

27. Prove that the most economical section of copper to employ in a transmission line is such that the interest on the capital invested in the copper is equal to the cost of the energy consumed in resistance, no other costs being supposed to influence the choice.

If the cost of copper is £100 per ton, the interest on the capital invested 6 per cent, the cost of electrical energy 0·75 pence per kilowatt hour, the density of copper 9 gm./cm.^{3}, the specific resistance of copper $1{\cdot}6 \times 10^{-6}$ ohm-cm. units, calculate the most economical section of conductors to carry 500 amperes to an outlying station for 12 hr. per day throughout the year.

(*Mech. Sc. Trip. 1912.*)

28. A smooth hemispherical bowl of radius a contains a rod of n times its weight, and is suspended from a point on its rim. Show that, if in the position of equilibrium the rod lies entirely within the bowl, it must be horizontal. Prove also that the inclination of the plane of the rim to the horizontal is $\tan^{-1}2(n + 1)$.

[The centroid of a hemispherical bowl bisects the radius to its vertex.] (*B.Sc. Pass, Lond. 1914.*)

29. A motor-car of mass m is driven by a constant force at all speeds, the resistance varying as the square of the velocity. Prove that the time taken to get up a velocity v from rest is

$$\frac{mv_0^2}{2P}\log_e\left(\frac{v_0+v}{v_0-v}\right),$$

where v_0 is the full speed and P is the power required to maintain this speed against the resistance.

A loaded car of 15 h.p. weighs 2000 lb., and its full speed is 45 m.p.h.; prove, on the hypothesis stated above, that it can attain a speed of 30 m.p.h., starting from rest, in a little under half a minute.

[It may be assumed that $g = 32$ in foot-second units and that $\log_e 10 = 2\cdot303$.]

(*Inter. B.Sc. Hons. Lond. 1913.*)

30. Show how a ballistic galvanometer may be used to measure flux, and how the constant in such a case may be determined.

A ballistic galvanometer was connected in series with the secondary of an electromagnetic flux standard and a current of 2 amperes was reversed in the primary of the standard.

The first fling observed in the galvanometer scale was 65 divisions, and the tenth swing on the same side 12 divisions.

The details of the flux standard were:

Turns in primary coil 500, length 40 cm., diameter 8 cm.

Turns in secondary coil 500, diameter 7·5 cm.

Find the change of flux per scale division.

If the resistance of the secondary circuit were halved, what would be the observed fling, assuming no damping effect due to friction? (*Mech. Sc. Trip. 1912.*)

31. By considering the air enclosed in a cylinder fitted with a piston, prove that the work done by the air in expanding from a volume v_1 to a volume v_2 is

$$\int_{v_1}^{v_2} p\,dv, \text{ where } p \text{ is the pressure per unit area.}$$

If a quantity of air expands in such a manner that no heat is lost or gained, then $p \times v^c$ remains constant, where $c = 1 \cdot 41$. Prove that the work done when the initial pressure and volume are p_1 and v_1 and the final pressure and volume are p_2 and v_2 is

$$\frac{1}{c-1}(p_1v_1 - p_2v_2).$$

Find p_2 and the work done if $v_1 = 10$ c. in., $v_2 = 20$ c. in., and $p_1 = 15$ lb. per square inch.

(*Army Entrance Exam. 1918.*)

32. Show that in an atmosphere of uniform temperature the pressure, p, at a height z above the ground, is given by the equation

$$p = p_0 e^{-\mu z},$$

where $\mu = \dfrac{g\rho_0}{p_0}$ and ρ_0 and p_0 are the density and pressure of the air at $z = 0$.

A hollow gas-tight sphere containing hydrogen requires a force mg to prevent it from rising when the lowest point touches the ground; the total mass of sphere and hydrogen is M. Show that the sphere can float in equilibrium with its lowest point at a height h above the ground, where

$$h = \frac{1}{\mu}\log_e \frac{M+m}{M}.$$

(*Maths. Trip. I. 1915.*)

33. If a drop of water is confined in a vessel, show that the increase in vapour pressure due to surface tension, in the state of equilibrium, is given by

$$\delta p = \frac{2\rho}{\sigma - \rho}\frac{T}{a},$$

where δp is the increase in pressure, σ the density of water and ρ of its vapour, T the surface tension, and a the radius of the drop.

Show that the corresponding change in vapour density arising from the electrification of the drop is given by

$$-\frac{1}{R\theta}\frac{\rho}{\sigma-\rho}\frac{e^2}{8\pi\kappa a^4},$$

where e is the electric charge, κ the dielectric constant, R the gas constant, and θ the temperature.

What bearing has this result on the size of raindrops we may expect to accompany thunderstorms?

[Assume the potential energy of a sphere carrying a charge e is given by $\frac{e^2}{2\kappa a}$.]

34. Show that the efficiency of the theoretical Diesel cycle is equal to

$$1-\frac{1}{r^{\gamma-1}}\left\{\frac{\rho^\gamma-1}{\gamma(\rho-1)}\right\},$$

where γ is the ratio of the specific heats and r and ρ the ratios of the maximum volume and cut-off volume respectively to the clearance volume.

Taking $\gamma = 1 \cdot 4$, find the value of this efficiency for an engine in which $r = 14$ and $\rho = 1 \cdot 6$. Find also the efficiency of the standard engine of comparison for the same compression ratio. (*Mech. Sc. Trip. 1914.*)

35. Show how to find the dryness fraction at any point in the adiabatic expansion of a liquid and its vapour, the condition of the mixture being known at one point of the expansion. The specific heat may be assumed constant.

If the dryness fraction remains constant during adiabatic expansion, show that the relation between the latent heat and the temperature must be of the form

$$T = ae^{-\frac{qL}{\sigma T}},$$

where a is a constant and σ is the specific heat of the liquid.

[L is the latent heat, T the temperature, and q the dryness fraction.] (*Mech. Sc. Trip. 1913.*)

36. In the case of a perfect gas evaluate the definite integral $\int \frac{dH}{T}$ between any two states defined by v_1, t_1 and v_2, t_2, where dH is the heat received per pound of the gas at absolute temperature T, irreversible processes being excluded.

Also, if the relation between pressure and volume is given by the equation pv^n = constant, show that the total heat received is proportional to $T_2 - T_1$, and find the complete expression for it.

(*Mech. Sc. Trip. 1909.*)

37. Show by thermodynamical reasoning that if L is the latent heat of vaporization at absolute temperature T of a substance, p the pressure, V the specific volume of the saturated vapour, and ω that of the liquid, then

$$\frac{(V - \omega)}{J} = \frac{L}{T}\left(\frac{\partial T}{\partial p}\right)_v,$$

where $\left(\frac{\partial T}{\partial p}\right)_v$ is the rate of increase of temperature with pressure, the volume being kept constant.

38. It is known that the discharge of electrons by hot platinum in a vacuum follows the law

$$i = DT^{\frac{1}{2}}e^{-\frac{P}{2T}},$$

where i is the saturation thermionic current per square centimetre of platinum surface, T the absolute temperature of the platinum, P the work which must be done in enabling as many electrons to escape from the metal as there are molecules in 1 gm. molecule of platinum, and D is a constant.

Show that if the emission of the negative electrons be assumed to be analogous to the evaporation of a liquid, the result of example 37 leads directly to the experimental law. (*H. A. Wilson.*)

39. Show thermodynamically that the electromotive force of a reversible electric cell is given by

$$E = \lambda + T\left(\frac{\partial E}{\partial T}\right),$$

where E is the electromotive force, λ the heat evolved or absorbed by the chemical changes accompanying the passage of unit quantity of electricity through the cell, and T is the absolute temperature.

40. Show on thermodynamic grounds that the surface energy of a liquid is not equal to S, the surface tension, but to $S - T\frac{\partial S}{\partial T}$. (*B.Sc. Hons. Lond. 1914.*)

The Thermo-electric Circuit.

Consider a circuit formed of a copper and an iron wire soldered together into a ring.

It is well known that if one junction is heated, a current of electricity will flow in the circuit. This current is called a "thermo-electric" current.

The work done in carrying the electric charges round the circuit is provided by the excess of the heat absorbed by the circuit over that liberated by it.

The absorption and liberation of heat takes place in two regions:

(*a*) At the junctions.
(*b*) In the metals.

Peltier (1834) discovered effect (*a*). The Peltier law is that the absorption (or liberation) of heat is proportional to the quantity of electricity that has passed the junction. The constant of proportionality (the *Peltier coefficient*) depends on the temperature of the junction and the sign of the heat *absorption* on the direction of the flow of electricity at the junction, i.e.

Heat liberated per second in passing from B to A at temperature T_1 ... ... $= \Pi_1 i$.
Heat absorbed per second in passing from A to B at temperature T_2 ... ... $= -\Pi_2 i$.

Kelvin worked out the second effect. He supposed electricity to possess a *specific heat* σ, different for different metals, so that, if σ_A is the specific heat of electricity in metal A, the heat absorbed in moving a quantity of electricity (e) from a point at temperature T to one at temperature $(T + 1)$ is $\sigma_A e$.

Both the Peltier and Thomson Effects are Reversible.

By considering the passage of a quantity of electricity e round the thermo-electric circuit, and applying the thermodynamic laws to the reversible interchanges of work and heat, we can arrive at the relationship of Π and σ to the temperature and electromotive force of the circuit.

41. Show by thermodynamical reasoning that on Kelvin's hypothesis of "the specific heat of electricity" the Peltier effect P at a thermojunction is equal to $T\frac{dE}{dT}$, and the difference of the specific heats of electricity in the two metals of the circuit is $-T\frac{d^2E}{dT^2}$, where T is the absolute temperature and E the electromotive force of the circuit.

(*B.Sc. Hons. Lond. 1914.*)

42. A volume of an ionized gas loses its ions by diffusion and by recombination.

If the effect of diffusion is negligible in a particular experiment, and if there are, initially, as many positive as negative ions, show, by the principle of mass action, that

$$\frac{1}{n_2} - \frac{1}{n_1} = \theta t,$$

where n_1 is the initial number of ions,
n_2 the final number of ions,
t the lapse of time,
and θ a constant.

This constant is called the *coefficient of recombination*, and it has been experimentally determined. It is nearly constant for air, oxygen, hydrogen, and carbon dioxide.

43. There is a method of determining the number of

molecules which take part in a chemical reaction, which depends on measuring the velocity of the reaction at widely different concentrations.

The formation of cyamelide from cyanic acid takes place according to the equation

$$n\text{CNOH} = \text{C}_n\text{N}_n\text{O}_n\text{H}_n.$$

An experiment showed the following results:

The concentration of the cyanic acid changed from 188·84 to 153·46 in 23 hr., the volume being v. It changed from 79·01 to 76·04 in 20 hr., the volume being V.

Show that

$$n = \frac{\log_e\left(\frac{dc}{dt}\right)_v - \log_e\left(\frac{dc}{dt}\right)_V}{\log_e c_v - \log_e c_V},$$

and hence deduce that the reaction is a trimolecular one.

44. Deduce the differential equation of mass action for a trimolecular reaction in the form

$$\frac{dx}{dt} = \kappa(A - x)(B - x)(C - x),$$

where A, B, and C are constants and x is the concentration of one of the reacting substances. Interpret these constants, and show that the integral of the equation is

$$\kappa = \frac{1}{t(C - A)}\left\{\frac{1}{A - B}\log_e\frac{B(A-x)}{A(B-x)} + \frac{1}{C - B}\log_e\frac{C(B-x)}{B(C-x)}\right\}.$$

45. If A = B = C, the solution in example 44 breaks down. Show that then

$$\kappa = \frac{x(2A - x)}{2A^2(A - x)^2 t}.$$

Experiments were made on the reaction between ferrous chloride, hydrochloric acid, and potassium chlorate at 20°C.

The quantity x of ferrous chloride used up was measured by titration with permanganate solution.

The hydrochloric acid was decinormal, and the ferrous chloride and potassium chlorate of corresponding strength, so that A = B = C = 0·1.

The quantitative reaction is

$$6\,FeCl_2 + KClO_3 + 6\,HCl = 6\,FeCl_3 + KCl + 3\,H_2O.$$

The reaction proceeded as follows:

Time in Minutes.		$10x$.
0	....	0·000
5	....	0·048
15	....	0·122
35	....	0·238
60	....	0·329
110	....	0·452
170	....	0·525

Show that the reaction is trimolecular and κ is nearly unity.

How do you reconcile this result with the chemical equation stated above?

46. Consider the formation of ethyl-acetate from ethyl-alcohol and acetic acid at 25° C.

$$\underset{\text{(acetic acid)}}{C_2H_4O_2} + \underset{\text{(alcohol)}}{C_2H_5OH} = \underset{\text{(ethyl-acetate)}}{C_4H_8O_2} + \underset{\text{(water)}}{H_2O}.$$

Initially the concentrations were, in Knoblauch's experiment:

$$C_{\text{acid}} = 1,$$
$$C_{\text{water}} = C_{\text{alcohol}} = 12·756.$$

During the initial stages of the reaction

$$\frac{\Delta C_{\text{ethyl-acetate}}}{\Delta t} = 0·00303 \text{ mols per litre per minute.}$$

Find the reaction constant (κ_1), assuming it is a di-molecular reaction.

47. The reaction in example 46 is reversible, and can be studied by starting with a mixture of ethyl-acetate and water, and measuring the rate of formation of acid.

In Knoblauch's test, he started with

$$C_{\text{ethyl-acetate}} = 1,$$
$$C_{\text{alcohol}} = C_{\text{water}} = 12 \cdot 215.$$

The experiment gave

$$\frac{\Delta C_{\text{acid}}}{\Delta t} = 0 \cdot 000996 \text{ mols per minute.}$$

Calculate the reaction constant (κ_2).

48. The reaction in example 46 was allowed to go on until equilibrium was established, and no further increase in ethyl-acetate took place. It was then found that

$$C_{\text{ethyl-acetate}} = 0 \cdot 7144 \text{ mols.}$$

Calculate the final concentrations of the acid, water, and alcohol.

Show that

$$\frac{C_{\text{ethyl-acetate}} \times C_{\text{water}}}{C_{\text{acid}} \times C_{\text{alcohol}}} = 2 \cdot 84.$$

Check this result with the experimental data of (47).

49. Experiments were made on the conversion of dibrom-succinic acid, in aqueous solution,

$$C_4H_4O_4Br_2 = C_4H_3O_4Br + HBr,$$

a monomolecular reaction.

The reaction constant was measured at different temperatures with the following results:

T° C.		κ (Time being in Minutes).
15	....	0·00000967
40	....	0·0000863
80	....	0·0046
101	....	0·0318

Show that these results are consistent with a formula

$$\log_{10}\kappa = a + bT.$$

What conclusions do you draw as to the general effects of temperature in promoting chemical reactions?

50. Cane-sugar, when treated with acidulated water, changes into two sugars of the same composition, but of different optical properties. Cane-sugar rotates a beam of polarized light in the right-handed direction; the "inverted" sugars are, on the whole, laevo-rotatory.

The reaction is expressed by the chemical equation,

$$\underset{\text{(cane-sugar)}}{C_{12}H_{22}O_{11}} + \underset{\text{(water)}}{H_2O} = \underset{\text{(dextrose)}}{C_6H_{12}O_6} + \underset{\text{(laevulose)}}{C_6H_{12}O_6},$$

and its progress can be followed with the polarimeter.

The reaction is monomolecular, as the water is greatly in excess, and the concentration of it is nearly constant. In one set of experiments, the reaction constants were

κ at 25° C., 0·765 mol per minute.
κ at 55° C., 35·5 mols per minute.

Show that the time taken for the concentration of the cane-sugar to fall to $\frac{1}{10}$ of its initial value is 3·01 min. at 25° C. and 0·0648 min. at 55° C.

NOTE.—Examples 43–50 inclusive are based on data given in Van 't Hoff's *Lectures on Theoretical and Physical Chemistry* (Arnold, 1898).

BIOGRAPHICAL NOTES

(*See Portraits*)

Napier, John (1550–1617), was born at Merchiston—now a part of Edinburgh.

He lived through the years of the Reformation, and took a prominent part as a zealous protestant in the religious and political discussions of his day. He wrote a commentary on the Book of Revelation with propositions and mathematical demonstrations, in which proposition 21, for example, is a proof that the Pope is Anti-Christ, and proposition 36 shows that the apocalyptical "locusts" are the Turks and Mohammedans.

There was a craze in his day for enormously long numerical calculations. One contemporary, for instance, practically devoted his life to finding a numerical approximation to the value of π, and finally obtained it correct to 35 places of decimals—in fact, the character Sir Walter Scott gives to David Ramsay in the *Fortunes of Nigel* is exactly that of a typical sixteenth century mathematician.

Napier was impressed by the enormous labour of these calculations, and set about trying to find a means for simplifying them. The result was his invention of logarithms, which seems all the more marvellous when one bears in mind that Napier had no index notation to help him. The index notation was invented years afterwards by Descartes.

Napier thought of the relative motion of two points connected with one another in the following way: as one point moved in a straight line with uniform velocity, the second point moved in another straight line with an accelerated velocity, so that as one point moved in a series of steps in arithmetical progression, the other point would move in a series of steps in geometrical progression. He put these ideas into numbers and developed his system of logarithms.

Kepler, Johann (1571–1630), a German astronomer, was born near Stuttgart.

His main contribution to astronomy is the group of three laws known as Kepler's Laws. He analyzed an enormous mass of astronomical observations, and discovered empirically the three laws in accordance with which the planets revolve round the sun. Newton showed that his law of gravitation was the *only* one consistent with Kepler's Laws.

Ill-luck dogged him all his life. He first succeeded Tycho Brahe as astronomer to Rudolph II, Emperor of the Holy Roman Empire, but this empire proved bankrupt and could not pay him his wages. Then his wife lost her reason and died; his second marriage was also unfortunate; and, finally, he was deprived of an appointment he held for heresy, and narrowly escaped with his life.

He cast horoscopes and told fortunes, for which he charged heavily; but his conscience seems to have pricked him for imposing on the credulity of his age, for he wrote: "Nature, which has conferred upon every animal the means of existence, has designed astrology as an adjunct and ally to astronomy".

Out of it all, however, he gave the world the laws of planetary motion and the idea of the integral calculus.

Descartes, René (1596–1650), a French mathematician and philosopher, and a contemporary of Galileo, was born near Tours.

As a lad he suffered from ill-health. "On account of his delicate health he was permitted to lie in bed till late in the mornings; this was a custom which he always followed, and when he visited Pascal in 1647 he told him that the only way to do good work in mathematics and to preserve his health, was never to allow anyone to make him get up in the morning before he felt inclined to do so; an opinion which I chronicle for the benefit of any schoolboy into whose hands this work may fall."[1]

He became a soldier by profession, and he is said to have thought of the method of co-ordinate geometry—Cartesian Geometry, as it is now named after him—in three dreams which he dreamt while campaigning on the Danube.

He does not appear to have been a particularly amiable man. His appearance was forbidding, and his personality seems to have been even more forbidding than his appearance. He despised learning and art, except for their utility.

He is celebrated in mathematics for his invention of co-ordinate geometry, and in philosophy for his dictum, "Cogito, ergo sum"; a dictum which called forth from Carlyle the comment: "*Cogito, ergo sum.* Alas, poor Cogitator, this takes us but a little way. Sure enough, I am; and lately was not: but Whence? How? Where to? The answer lies around, written in all colours and motions, uttered in all tones of jubilee and wail, in thousand-figured, thousand-voiced, harmonious Nature."

Sir Isaac Newton (1642–1727), the English mathematician and natural philosopher, was born near Grantham in Lincolnshire on Christmas Day.

His father was a yeoman farmer, and it was intended that Isaac should carry on the farm. At the age of fourteen he showed distinct mechanical ability, and his widowed mother decided to give him a chance to develop his mechanical instincts. He was sent to Trinity College, Cambridge, and went through the ordinary college course

[1] Ball's *History of Mathematics.*

in classics and mathematics. He accidentally came across a book on astrology which interested him, but he could not understand it on account of the geometry and trigonometry in it. So he bought a copy of Euclid, and was surprised to find how plain it all seemed.

On account of the great plague, he left Cambridge in 1665–6 and went to live at his home in Grantham. It was during this visit that his ideas on gravitation took definite shape. He also invented the differential calculus about this time.

In 1669 he was appointed Lucasian professor in Cambridge, and gave much of his time to the study of optics. His writings were attacked by many of his contemporaries with an arrogant confidence born of ignorance, and Newton had an uphill task from 1672 to 1675, and more than once determined to give up philosophy except for his own satisfaction. "I see I have made myself a slave to philosophy; but if I get rid of Mr. Linus's business, I will resolutely bid adieu to it eternally, excepting what I do for my private satisfaction, or leave to come out after me; for I see a man must either resolve to put out nothing new, or to become a slave to defend it."

Of Newton's absent-mindedness many stories have been told. A typical one narrates how, when he was entertaining some friends to dinner and went to draw some wine, he stayed away so long that his friends set off to seek him. They found him in the wine cellar, busy solving a mathematical problem, his friends forgotten, and the jug still unfilled.

He was no "eight hours' day" man, and would often spend eighteen or nineteen hours out of the twenty-four at the most exhausting kind of work.

Newton had the strongest sense of duty. When he was appointed Master of the Mint, he stoutly refused to engage in any of his favourite scientific pursuits, as he felt that his whole energies should be given to the service of the State. He broke away from this rule once, for

Leibnitz apparently was too much for his patience. The German mathematician had been working at a set of problems and was unable to solve them. He thereupon concluded they were insoluble, and published them as a challenge to the world. Newton received the problems one evening on his return from the mint, and at the end of five hours had solved them all.

He spent nearly half his time studying chemistry and theology. Once when Halley made some flippant remark on a question of religion, Newton fell upon him with the rebuke: "I have studied these things: you have not!"

His greatest contributions to physical science were his theory of gravitation, by which he accounted for the motions of the bodies of the solar system, and of the moons around the planets; and the differential calculus. He made many other important, though minor, contributions to mathematics and physics. Here is Newton's own estimate of himself:

"I do not know what I may appear to the world; but to myself I seem to have been only like a boy, playing on the sea shore, and diverting myself, in now and then finding a smoother pebble, or a prettier shell than ordinary, while the great ocean of truth lay all undiscovered before me."

Leibnitz, Gottfried Wilhelm (1646–1716), one of the most brilliant men of his time, was born at Leipzig. He was educated first at Leipzig University, which refused him the degree of Doctor of Laws, on account of his youth. He thereupon removed to Nuremberg, where he continued the study of the law and subsequently entered the diplomatic service.

In 1674 Leibnitz became attached to the Court of Brunswick, and from that time he had greater leisure to devote to his favourite pursuits of mathematics and philosophy. He propounded his views on the differential and integral calculus between 1674 and 1677, and an acrimonious debate ensued among contemporary mathematicians as to who was really the inventor of the differential calculus, Newton or Leibnitz. The dispute seems to have been largely a matter of words. Newton appears to have discovered for himself the view of the differential calculus as "a rate of growth", and he expressed a rate of growth or a *fluxion*, as he called it, by putting a dot over the letter, just as is done in dynamics at the present day. The dot is satisfactory enough when "time" is the independent variable, and when the rate of growth is a speed, but it is not very convenient when other independent variables are used. Newton used a method akin to the modern method of limits to arrive at these fluxions. Leibnitz, on the other hand, invented the dy/dx notation, which is used to-day. This notation is not altogether free from objection. It suggests first of all that the quantity represented by dy/dx is a fraction equal to dy divided by dx, and secondly, the first thing the beginner wants to do is to cancel the d's. However, Leibnitz's notation appears to be now standardized, except in dynamics, where the Newtonian notation with dots is still commonly used—and convenient. Leibnitz's reputation rests as much on his philosophical as on his mathematical work. He was the propounder of a system of philosophy known as Monadology. It is curious that the practical experimenter, Newton, developed the calculus on more philosophical lines than Leibnitz, whereas the philosopher, Leibnitz, gave the world the practical notation which is used to-day.

Volta, Alessandro (1745–1827), an Italian physicist, is chiefly celebrated as a pioneer in electrical science. He was born at Como. The practical unit of electrical pressure—the volt—is named after him.

Faraday, Michael (1791–1867), an English physicist and chemist, was born at Newington, Surrey, of Yorkshire stock, and was the son of a blacksmith. He was apprenticed to a bookbinder, and educated himself.

He was "discovered" by Sir Humphry Davy, who made him a laboratory assistant at the Royal Institution, and took him on a tour through France, Italy, and Switzerland. He occupied different posts at the Royal Institution, and finally became professor of chemistry there.

He was an experimenter of the highest skill, and possessed great powers of physical reasoning. Sir J. J. Thomson says: "Faraday, who possessed, I believe, almost unrivalled mathematical insight, had had no training in analysis, so that the convenience of the idea of action at a distance for purposes of calculation had no chance of mitigating the repugnance he felt to the idea of forces acting far away from their base and with no physical connection with their origin". His reputation rests chiefly on his electrical discoveries. He would have nothing to do with "action at a distance", and insisted on the "axiom" that matter cannot act except where it is. This led him to picture space as filled with a medium—ether—and to ascribe the electric and magnetic field to states of stress and strain in this medium. He pictured the medium as filled with "tubes of electric and magnetic force", and proved that the phenomena of electromagnetic induction—which he had discovered—could be accounted for as an effect produced by a change in the number of "Faraday Tubes" passing between the two circuits concerned. The same theory accounted for the induction of currents in closed circuits by magnets in motion relative to them.

Carnot, Sadi Nicolas Léonard (1796–1832), was a French army officer and a contemporary of Fourier.

He was the son of an eminent geometrician. In 1824 he published a paper "Réflexions sur la puissance motrice du feu", which laid the foundations of modern thermodynamics. Though his paper is based on what is now considered a wrong notion, namely, that heat is a material substance, only a slight modification is needed to bring his work into line with modern views on this point. His methods are simple, and yet most original and profound, and they form the basis of all later work on the relations between heat and mechanical energy.

Henry, Joseph (1797–1878), an American physicist, was born in Albany, N.Y. He made several practical electrical inventions, and carried out important experiments on electrical induction. The practical unit of electromagnetic induction—the Henry—is named after him.

Graham, Thomas (1805–69), a Scottish chemist, was born in Glasgow. He was the son of a merchant, and was educated at Glasgow University. He gave special study to the subject of molecular physics, and made important contributions to our knowledge of the diffusion of gases and of the properties of colloids. The subject of colloids is becoming important because of its industrial applications.

Joule, James Prescott (1818–89), an English physicist, was born at Manchester.

His chief scientific work consisted in the proof of the relation between mechanical energy and heat. He demonstrated that if mechanical energy disappears in a system of bodies and no other physical change takes place except a change in the temperatures of the different bodies of the system, then for every 778 (roughly) foot-pounds of mechanical energy that disappear, 1 B.Th.U. of heat makes its appearance. Thus heat and mechanical work are interchangeable forms of physical energy.

Clausius, Rudolf Julius Emmanuel (1822–88), a German physicist, was one of the pioneers of the science of thermodynamics. He was a contemporary of Lord Kelvin, and these two physicists covered much the same ground in thermodynamics by different methods. Clausius also did very important work in the kinetic theory of gases and in the theory of electrolysis.

Kelvin, Lord (Sir William Thomson) (1824–1907), was born in Belfast. His father was Professor James Thomson, an Irishman, and his mother, Margaret Gardiner, the daughter of a Glasgow merchant. The family settled in Glasgow in 1832, and William Thomson was educated at Glasgow University and later at Peterhouse, Cambridge.

Lord Kelvin's main scientific work consisted in the contributions he made to the theory of the conservation of energy. He lived in an age of great mechanical developments. Watt's steam-engine had been invented before he was born, but the theory underlying its action was only beginning to be understood about 1840. Electrical developments were taking place rapidly, and these industrial applications of electric power depended, in their scientific aspects, mainly upon the principle of conservation of energy. He applied himself to thermodynamics, and did more perhaps than any other man to place the theory on its present basis. He also laid the foundations for the exact measurement of electrical quantities, and so brought the science of electricity and magnetism within the range of exact treatment. In the realm of theoretical physics, Lord Kelvin made important contributions to the theories of hydrodynamics and elasticity. He also constructed a hydrodynamical theory of the constitution of matter, based on vortex atoms in an infinite fluid. In collaboration with Professor Tait of Edinburgh he wrote a *Treatise on Natural Philosophy*, which entirely changed the standpoint from which the subject was viewed in this country. He took part in the laying of the first submarine cable to America, and acted as adviser in many engineering and industrial enterprises.

Clerk Maxwell, James (1831–79), a Scottish mathematician and physicist, was born in Edinburgh. He was educated at Edinburgh Academy, Edinburgh University, and Cambridge. He was Professor of Natural Philosophy in Aberdeen from 1856 to 1860, and subsequently became Cavendish Professor of Physics at Cambridge.

Working on Faraday's physical ideas of the electromagnetic field, he investigated them mathematically with the highest skill, and showed that an electromagnetic theory of light could also be based on the Faraday-Maxwell equations. These equations are still the "last word" in electrical theory, and are the starting point of the modern theory of relativity, which centres round Einstein's name.

He was a man of great intellectual power. P. G. Tait writes in the *Encyclopædia Britannica*: "In private life Clerk Maxwell was one of the most lovable of men, a sincere and unostentatious Christian. Though perfectly free from any trace of envy or ill-will, he yet showed on fit occasion his contempt for that pseudo-science which seeks for the applause of the ignorant by professing to reduce the whole system of the universe to a fortuitous sequence of uncaused events."

Guldberg, Cato Maximilian (1836–1902), a Norwegian mathematician and physicist, was born at Christiania. He was the eldest son of Carl August Guldberg, a clergyman.

He was educated in the Royal Norwegian University at Christiania, and subsequently studied in France, Switzerland, and Germany. In 1869 he was appointed Professor of Applied Mathematics in the Royal University, a position which he held until his death. He devoted himself from his student days to the study of chemical dynamics. At the age of twenty-eight he contributed a paper (in association with Professor Waage), on the Law of Mass Action, to the Society of Science in Christiania. This was followed by a second paper in 1867 entitled: "Études sur les affinités chimiques", and by a third paper in 1879, in which the theory of Mass Action was given in its final form. He contributed many papers to learned societies on Physical Chemistry. He was a keen hunter and fisherman, and was for a long time President of the Norwegian Hunting and Fishing Society.

Gibbs, Josiah Willard (1839–1903), an American mathematical physicist, was born in New Haven, Connecticut, U.S. He was educated at Yale, and later became professor of mathematical physics there. He studied specially the applications of thermodynamics to chemistry, and has been called "the founder of chemical energetics". He wrote important papers on *Vector Analysis*. His name is especially associated with a theorem in physical chemistry known as the "Phase Rule".

Van 't Hoff, Jacobus-Hendrikus (1852–1911), a Dutch chemist, was born at Rotterdam. He was the founder, with Le Bel, of *Stereochemistry*, or the chemistry of space, in which the formula of the substance is represented not in one plane, as the usual graphical formulæ are, but in the solid; for instance, CH_4 is represented by a carbon atom at the centroid of a tetrahedron with a hydrogen atom at each corner. He also did a great deal of work on chemical dynamics and the law of Mass Action.

The reader who is interested in the history of mathematics and its devotees, should read *A Short Account of the History of Mathematics* by W. W. Rouse Ball.

TABLE I

LIST OF INDEFINITE INTEGRALS

1. $\int x^n dx = \frac{x^{n+1}}{n+1}$, unless $n = -1$.

2. $\int \frac{dx}{x} = \log_e x$, or $\log_e(-x)$. 3. $\int e^x dx = e^x$.

4. $\int \sin x dx = -\cos x$. 5. $\int \cos x dx = \sin x$.

6. $\int \sec^2 x dx = \tan x$. 7. $\int \frac{dx}{\sqrt{(a^2 - x^2)}} = \text{arc}\sin\frac{x}{a}$.

8. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\text{arc}\tan\frac{x}{a}$.

9. $\int \frac{dx}{\sqrt{(x^2 + k)}} = \log_e(x + \sqrt{x^2 + k})$.

10. $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\log_e\frac{x - a}{x + a}$, if $x^2 > a^2$

$= \frac{1}{2a}\log_e\frac{a - x}{a + x}$, if $x^2 < a^2$.

11. $\int \sqrt{(a^2 - x^2)}dx = \frac{1}{2}x\sqrt{(a^2 - x^2)} + \frac{1}{2}a^2 \text{arc}\sin\frac{x}{a}$.

12. $\int \sqrt{(x^2 + k)}dx = \frac{1}{2}x\sqrt{(x^2 + k)} + \frac{1}{2}k\log_e(x + \sqrt{x^2 + k})$.

13. $\int e^{ax}\cos bx\, dx = \frac{e^{ax}}{a^2 + b^2}(a\cos bx + b\sin bx)$.

14. $\int e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2 + b^2}(a\sin bx - b\cos bx)$.

15. $\int \frac{dx}{\sin x} = \log_e \tan\frac{1}{2}x$. 16. $\int \frac{dx}{\cos x} = \log_e \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$.

17. $\int \sin^2 x\, dx = \int \frac{1}{2}(1 - \cos 2x)dx = \frac{1}{2}x - \frac{1}{4}\sin 2x$.

18. $\int \cos^2 x\, dx = \int \frac{1}{2}(1 + \cos 2x)dx = \frac{1}{2}x + \frac{1}{4}\sin 2x$.

These are all standard integrals which are constantly occurring in the applications of the calculus. The first ten at least should be committed to memory.

TABLE II

NATURAL LOGARITHMS

x	$\log_e x$	x	$\log_e x$	x	$\log_e x$	x	$\log_e x$
1·1	·0953	3·4	1·2238	5·7	1·7405	8·0	2·0794
1·2	·1823	3·5	1·2528	5·8	1·7579	8·1	2·0919
1·3	·2624	3·6	1·2809	5·9	1·7750	8·2	2·1041
1·4	·3365	3·7	1·3083	6·0	1·7918	8·3	2·1163
1·5	·4055	3·8	1·3350	6·1	1·8083	8·4	2·1282
1·6	·4700	3·9	1·3610	6·2	1·8245	8·5	2·1401
1·7	·5306	4·0	1·3863	6·3	1·8405	8·6	2·1518
1·8	·5878	4·1	1·4110	6·4	1·8563	8·7	2·1633
1·9	·6419	4·2	1·4351	6·5	1·8718	8·8	2·1748
2·0	·6931	4·3	1·4586	6·6	1·8871	8·9	2·1861
2·1	·7419	4·4	1·4816	6·7	1·9021	9·0	2·1972
2·2	·7885	4·5	1·5041	6·8	1·9169	9·1	2·2083
2·3	·8329	4·6	1·5261	6·9	1·9315	9·2	2·2192
2·4	·8755	4·7	1·5476	7·0	1·9459	9·3	2·2300
2·5	·9163	4·8	1·5686	7·1	1·9601	9·4	2·2407
2·6	·9555	4·9	1·5892	7·2	1·9741	9·5	2·2513
2·7	·9933	5·0	1·6094	7·3	1·9879	9·6	2·2618
2·8	1·0296	5·1	1·6292	7·4	2·0015	9·7	2·2721
2·9	1·0647	5·2	1·6487	7·5	2·0149	9·8	2·2824
3·0	1·0986	5·3	1·6677	7·6	2·0281	9·9	2·2925
3·1	1·1314	5·4	1·6864	7·7	2·0412	10·0	2·3026
3·2	1·1632	5·5	1·7047	7·8	2·0541	—	—
3·3	1·1939	5·6	1·7228	7·9	2·0669	—	—
10·5	2·3513	14·0	2·6391	17·5	2·8621	22·0	3·0911
11·0	2·3979	14·5	2·6740	18·0	2·8904	23·0	3·1355
11·5	2·4430	15·0	2·7081	18·5	2·9173	24·0	3·1781
12·0	2·4849	15·5	2·7408	19·0	2·9444	25·0	3·2189
12·5	2·5262	16·0	2·7726	19·5	2·9703	26·0	3·2581
13·0	2·5649	16·5	2·8034	20·0	2·9957	27·0	3·2958
13·5	2·6027	17·0	2·8332	21·0	3·0445	28·0	3·3322

TABLE III

EXPONENTIAL FUNCTIONS

x	e^x	e^{-x}	x	e^x	e^{-x}
·05	1·0513	·9512	·55	1·7333	·5769
·1	1·1052	·9048	·6	1·8221	·5488
·15	1·1618	·8607	·65	1·9155	·5220
·2	1·2214	·8187	·7	2·0138	·4966
·25	1·2840	·7788	·75	2·1170	·4724
·3	1·3499	·7408	·8	2·2255	·4493
·35	1·4191	·7047	·85	2·3396	·4274
·4	1·4918	·6703	·9	2·4596	·4066
·45	1·5683	·6376	·95	2·5857	·3867
·5	1·6487	·6065	1·0	2·7183	·3679
1·1	3·0042	·3329	3·1	22·198	·0450
1·2	3·3201	·3012	3·2	24·533	·0408
1·3	3·6693	·2725	3·3	27·113	·0369
1·4	4·0552	·2466	3·4	29·964	·0334
1·5	4·4817	·2231	3·5	33·115	·0302
1·6	4·9530	·2019	3·6	36·598	·0273
1·7	5·4739	·1827	3·7	40·447	·0247
1·8	6·0496	·1653	3·8	44·701	·0224
1·9	6·6859	·1496	3·9	49·402	·0202
2·0	7·3891	·1353	4·0	54·598	·0183
2·1	8·1662	·1225	4·1	60·340	·0166
2·2	9·0250	·1108	4·2	66·686	·0150
2·3	9·9742	·1003	4·3	73·700	·0136
2·4	11·023	·0907	4·4	81·451	·0123
2·5	12·183	·0821	4·5	90·017	·0111
2·6	13·464	·0743	4·6	99·484	·0101
2·7	14·880	·0672	4·7	109·95	·0091
2·8	16·445	·0608	4·8	121·51	·0082
2·9	18·174	·0550	4·9	134·29	·0074
3·0	20·086	·0498	5·0	148·41	·0067

ANSWERS TO EXERCISES

AND HINTS FOR SOLUTIONS OF THE MORE DIFFICULT PROBLEMS

Page 12

(i) A straight line passing through the points (0, 3), $(-\frac{3}{2}, 0)$. (ii) A circle of radius 2 units. (iii) The curve lies entirely on the right-hand side of Y′OY, and is symmetrical about OX. It passes through the origin and is called a parabola.

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1. 8. **2.** 219. **3.** $1 + 4a$. **4.** 1.

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1. 0·9696 c. in.; 432·1 c. in. per inch. **2.** 602.

Exercise 1

1. The lower limit A_1 is 16·50 sq. in. The upper limit A_2 is 21·50 sq. in. The difference is 30·3 per cent of A_1. **2.** A_1 is 18·15 sq. in.; A_2 is 20·65 sq. in. The difference is 13·8 per cent of A_1. The true area is 19·635 sq. in.

3.

Δx	0·1	0·01	0·001	0·0001	$\rightarrow 0$
Δy	1·21	0·1201	0·012001	0·00120001	$\rightarrow 0$
$\frac{\Delta y}{\Delta x}$	12·1	12·01	12·001	12·0001	$\rightarrow 12$

4.

Δx	0·1	0·01	0·001	$\rightarrow$ 0
Δy	19·441	1·922401	0·192024001	$\rightarrow$ 0
$\dfrac{\Delta y}{\Delta x}$	194·41	192·2401	192·024001	$\rightarrow$ 192

5. $4x^3$. **6.** (i) $5x^4$, (ii) $8x^7$, (iii) $11x^{10}$. **7.** 0·028 sq. in. per sec.

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63° nearly. The effect of the scale is to make $\tan\psi = 10\tan\phi$, if ϕ is the actual angle on the graph.

Exercise 2

1. $3x + 7y = 27$; $\tan\psi = -\frac{3}{7}$, i.e. $\psi = 156° 48'$. **2.** $y = \dfrac{1}{\sqrt{3}}x + (7 - \sqrt{3})$. **3.** $y = 2x + 13$. **4.** $\tan\psi = -\frac{3}{4}$, ∴ $\psi = 143° 8'$ nearly. **5.** $\dfrac{dy}{dx} = 4$, $y = 4(x - 1)$. **6.** $\dfrac{dv}{dt}$. **8.** 2·43 c. in. **9.** 5·41 m.p.h. **10.** 0·51 c. in.

Exercise 3

1. 2540, 3780; mean 3160. **2.** 3160. **3.** 3124. **4.** 116, 142; mean, 129; yes; that the integral of e^x is e^x. **5.** 1·98; yes; that the integral of $\sin x$ is $-\cos x$. **6.** 0·033. **7.** $s = \int v\,dt + c$, where c is a constant; $s = \int_0^T 3\mu t^2 dt = \mu\int_0^T 3t^2 dt = \mu T^3$. The distance travelled is proportional to the cube of the duration of the journey.

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1. If $y \rightarrow b$ when $x \rightarrow a$, then an η can be found so that $0 < |y - b| < \epsilon$ when $0 < |x - a| < \eta$ whatever ϵ may be, and $|y - b| = |(y + a) - (a + b)|$, whence result. **2.** Let $x = a + \delta$ where δ may be positive or negative. Then $x^3 - a^3 = \delta(3a^2 + 3a\delta + \delta^2)$, i.e. $|x^3 - a^3| < |\delta|M$, where M is the greatest numerical value of $3a^2 + 3a\delta + \delta^2$.

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1. 0·5236; 0·7854; 1·571. **2.** 57° 18′; 28° 39′; 88° 14′.

Exercise 4

1. a^2. **2.** $\sin a$. **3.** $\frac{1}{2}$. **4.** 0·7540; 0·9091; 1·8736. **5.** 19° 6′; 171° 53′; 315° 8′. **6.** If arc $\tan x = \theta$, then $\frac{\text{arc}\tan x}{x} = \frac{\theta}{\tan\theta}$. Now apply equation (12), p. 106. **9.** The error occurs in the fourth line. We divide by $(x - a)$ to get $(x + a) = a$ and $(x - a)$ is zero if $x = a$.

Exercise 5

1. Because acceleration is rate of change of velocity with time, while velocity itself is rate of change of distance or displacement with time, i.e. velocity is distance per second, and acceleration is velocity per second, i.e. (distance per second) per second. The accelerations are the same, for 1 mile per hour per second is $\frac{1}{3600}$ mile per second per second, and 1 mile per second per hour is $\frac{1}{3600}$ mile per second per second. **2.** 205 sec.; 1·95 miles; 34·39 m.p.h. **3.** At 25 m.p.h. $Pv = mfv$, hence

$$\text{H.P.} = \frac{1}{550}\left[\frac{70 \times 2240 \times 1\cdot25 \times 88 \times 25 \times 88}{32\cdot2 \times 60 \times 60}\right] = 594.$$

4. $F = \frac{2w}{\sqrt{3}}\int_0^{\frac{l\sqrt{3}}{2}} x^2dx = \frac{1}{4}wl^3$. **5.** $I = \int_0^r \rho\, 2\pi r dr\, r^2$ (ρ being the surface density) $= 2\pi\rho\int_0^r r^3dr = \frac{Mr^2}{2}$ where $M = \pi r^2\rho$.
6. Divide the plate into strips parallel to the axis, and integrate.
7. 74 lb.-in.^{2}. **8.** 224 lb.-in.^{2}. In the second case a large part of the total mass is between 5 and 6 in. from the axis, and hence contributes heavily to the moment of inertia, which depends on the square of the distance of the mass from the axis. **9.** 447 lb.-in.^{2}; 36·8 lb.-in.^{2}. **10.** The matter should be as far as possible from the axis about which the moment of inertia is to be taken. Yes. **11.** 85 % of the mass is more than 2′ 9″ from the axis and less than 3′ 0″ from the axis. Hence the moment of inertia of this part of the mass is less than $0\cdot85 \times 3^2 = 7\cdot65$ ton-ft.^{2}, and is more than $0\cdot85 \times 2\cdot75^2 = 6\cdot43$ ton-ft.^{2}. The remaining mass contributes moment of inertia less than $0\cdot15 \times 2\cdot75^2 = 1\cdot16$ ton-ft.^{2}, and more than zero. Hence max. I is 8·81 ton-ft.^{2} and min. I is 6·43 ton-ft.^{2}; mean I is 7·62 ton-ft. The error in this figure cannot exceed 15·6 %. **12.** 8·44 ton-ft.^{2}. **13.** $38\cdot82 \times 10^{27}$ ton-mile^2. **14.** $E - irl^2$.

Exercise 6

1. $7x^{\frac{4}{3}}$. 2. $3x^2 + 2x + 1$. 3. $2x + a + b$.

4. $1 + 3x^2 + 4x^3 + 6x^5$. 5. $-nx^{-n-1}$. 6. $\dfrac{b-a}{(x+b)^2}$.

7. $\dfrac{mx^{m-1}}{(x+1)^{m+1}}$. 8. $\dfrac{x}{\sqrt{1+x^2}}$. 9. $2m(2ax + x^2)^{m-1}(a + x)$.

10. $\dfrac{3x^2}{(1+x^6)^{\frac{3}{2}}}$. 11. $\sec^2 x$. 12. $\sec x \tan x$. (Rule 7.)

13. $-\operatorname{cosec} x \cot x$. (Rule 7.) 14. $-\operatorname{cosec}^2 x$. 15. $3\cos 3x$.

16. $\cos 3x \cos 2x + 2\cos 5x$. 17. $-\sin x \cos(\cos x)$. (Put $z = \cos x$.)

18. $\dfrac{1 + a\sin x + b\cos x}{(b + \cos x)^2}$. (Rule 7.) 19. $b\cos x - a\sin x + \cos^2 x - \sin^2 x$. (Rule 6.) 20. $\frac{1}{2}\dfrac{b\cos x}{\sqrt{a + b\sin x}}$. (Rules 2 and 8.)

21. $\dfrac{ax^4}{4}$. 22. $-\dfrac{a}{x}$. 23. $\dfrac{n}{m+n}ax^{\frac{m+n}{n}}$. 24. $\dfrac{(ax^n + b)^{m+1}}{na(m+1)}$.

25. $\dfrac{(ax+b)^{m+1}}{a(m+1)}$. 26. $-3\cos x$. 27. $\frac{1}{2}\sin^2 x$.

28. $\sec x$. (Ex. 12 above.) 29. $10\tan x$. (Ex. 11 above.)

30. $-\frac{1}{2}\cos 2x$.

Exercise 7

1. $I = \rho\int_0^l x^2 dx$ where ρ is the linear density; 163·3 lb.-ft.^{2}.

2. $0{\cdot}4244r$ from the centre of the circle along the axis of symmetry, r being the radius. 3. $\frac{3}{8}r$ from the centre of the sphere, along the axis of symmetry, r being the radius.

Example, p. 158

$\dfrac{1}{a}\arctan\dfrac{x}{a}$.

Examples, p. 162

1. (i) $-2\cos\sqrt{x}$ (put $z = \sqrt{x}$); (ii) $\sqrt{1+x^2}$ (put $z = 1 + x^2$); (iii) $\dfrac{x\sqrt{a^2-x^2}}{2} + \dfrac{a^2}{2}\arcsin\dfrac{x}{a}$ (put $x = a\sin\theta$, whence integral depends on $\int\cos^2\theta\, d\theta$; integrate this by parts); (iv) $\arcsin x$ (put $x = \sin\theta$); (v) $\arctan x$ (put $x = \tan\theta$); (vi) $\dfrac{1}{a}\operatorname{arc\,sec}\dfrac{x}{a}$ (put $x = a\sec\theta$). 2. (i) $x^2\sin x + 2x\cos x - 2\sin x$; (ii) $\dfrac{x}{2}\sin 2x + \frac{1}{4}\cos 2x$; (iii) $\dfrac{x^3}{3}\arcsin x + \dfrac{x^2}{9}\sqrt{1-x^2} + \frac{2}{9}\sqrt{1-x^2}$;

(iv) $-\sqrt{1-x^2}$ arc $\sin x + x$. **3.** Apply formula (7), p. 160.
4. By integration by parts make $\int \sin^p\theta \cos^q\theta\, d\theta$ depend on $\int \sin^{p-2}\theta \cos^q\theta\, d\theta$; $\sin^{p-1}\theta \cos^{q+1}\theta$ vanishes when $\theta = 0$ or $\frac{\pi}{2}$, hence $\int_0^{\frac{\pi}{2}} \sin^p\theta \cos^q\theta\, d\theta = \frac{p-1}{p+q} \int_0^{\frac{\pi}{2}} \sin^{p-2}\theta \cos^q\theta\, d\theta$; hence $\int_0^{\frac{\pi}{2}} \sin^4\theta \cos^2\theta\, d\theta = \frac{3}{6}\frac{1}{4} \int_0^{\frac{\pi}{2}} \cos^2\theta\, d\theta = \frac{3}{6}\frac{1}{4}\frac{1}{2}\Big[\sin\theta\cos\theta + \theta\Big]_0^{\frac{\pi}{2}}$
$= \frac{3}{6}\frac{1}{4}\frac{1}{2}\frac{\pi}{2}$.

Exercise 8a

1. $\frac{a}{2\sqrt{x}}$. **2.** $-\sin x \cos(\cos x)$. **3.** $\cos 3x \cos 2x + 2\cos 5x$.
4. $(1-2x-x^2)/(1+x^2)^2$. **5.** $\sin 2x$.
6. $\frac{1}{1+x^2}$. **7.** $a^2/(a^2-x^2)^{\frac{3}{2}}$. **8.** $\sqrt{\frac{1-x}{1+x}}$.
9. $\frac{n}{\cos^2 x + n^2 \sin^2 x}$. **10.** $\tan^4 x$. **11.** $\frac{1}{\sqrt{(1-x^2)}\,(1-x)}$.
12. $\frac{-3}{\sqrt{1-x^2}}$. **13.** $\frac{1}{\sqrt{(1-2x-x^2)}}$. **15.** $\frac{1}{2}\sqrt{(1+\operatorname{cosec} x)}$.
14. $(a+x)^{m-1}(b+x)^{n-1}\{m(b+x)+n(a+x)\}$. **16.** $\frac{nx^{n-1}}{(1+x)^{n+1}}$.
17. arc $\sin x + \frac{x}{\sqrt{1-x^2}}$. **18.** $\frac{1}{2}$. **19.** $2x$ arc $\tan\frac{x}{a} + a$.
20. $\sec^2 x$ arc $\tan x + \frac{\tan x}{1+x^2}$. **21.** $-\frac{2}{\sqrt{1-x^2}}$. **22.** arc $\tan\sqrt{\frac{x}{a}}$.
23. $\frac{2}{1+x^2}$. **24.** $\frac{1}{2(1+x^2)}$. **25.** $-\frac{2nx^{n-1}}{x^{2n}+1}$. **26.** $\frac{1}{1+x^2}$.
27. $\sin^2 x(3-4\sin^2 x)$. **28.** $\frac{1-x^2}{1+x^2}\frac{1}{\sqrt{1+x^4}}$. **29.** $a(\sin x + \cos x)$.
30. $\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{dy}{dz}\Big/\frac{dx}{dz}$. Express $\frac{dy}{dx}$ as a function of z, and then eliminate z by means of $x = a(1-\cos z)$.

Exercise 8b

1. $\frac{n}{m+n} a x^{\frac{m+n}{n}}$. **2.** $\frac{(ax+b)^{n+1}}{a(n+1)}$. **3.** $\frac{1}{\sqrt{6}}$ arc $\tan\left(x\sqrt{\frac{3}{2}}\right)$.

[Consider integral $\int \frac{1}{a + bx^2} dx = \frac{1}{a}\int \frac{dx}{1 + \frac{b}{a}x^2}$. Put $z = \sqrt{\frac{b}{a}}x$, hence $\frac{1}{a}\int \frac{dx}{1 + \frac{b}{a}x^2} = \frac{1}{\sqrt{ab}}\int \frac{dz}{1 + z^2}$. Now apply Ex. 1 (v), p. 162.]

4. $2(a + x)^{\frac{3}{2}}\left\{\frac{(a + x)^2}{7} - \frac{2a}{5}(a + x) + \frac{a^2}{3}\right\}$.

5. Remove brackets. Result is $\frac{ax^3}{3} + \frac{bx^4}{4}$. **6.** Assume $\frac{a^2}{x^2} + 1 = t^2$, whence integral is $\frac{x}{a^2\sqrt{a^2 + x^2}}$. **7.** $\frac{(ax^3 + b)^3}{9a}$.

8. $-\frac{\sqrt{(x^2 + 1)}}{x}$. **9.** $\tan x + \frac{\tan^3 x}{3}$ (put $\tan x = t$).

10. $\sin x - x \cos x$ (integrate by parts). **11.** $\frac{1}{2}(1 + x^2)$ arc $\tan x - \frac{1}{2}x$ (integrate by parts). **12.** $(a + bx + cx^2)^2/2$.

13. Result is $\frac{ax^4}{4} + \frac{bx^5}{5}$. **14.** $\frac{1}{2}\frac{1}{(1-x)^2} - \frac{1}{(1-x)}$.

15. $\frac{3x}{2} - 2 \sin x + \frac{\sin 2x}{4}$ (put $1 - \cos x = 2 \sin^2\frac{x}{2}$, or expand $(1 - \cos x)^2$ and integrate term by term). **16.** $\frac{(ax^n + b)^{m+1}}{na(m + 1)}$ (put $ax^n + b = z$). **17.** $\int \tan^{2n}\theta = \int \tan^2\theta \tan^{2n-2}\theta\, d\theta$, whence the result. $-(-1)^n$ means that the sign of $\tan\theta$ is negative if n is even and positive if n is odd. **18.** $\frac{x^3}{9} - \frac{2x}{9} + \frac{2\sqrt{2}}{9\sqrt{3}}$ arc $\tan x\sqrt{\frac{3}{2}}$. (Repeated integration by parts, the integral finally depending on question 3 above, or by division.) **19.** $x \tan\frac{x}{2}$.

20. $\left(x^3 + \frac{3x}{2}\right)\frac{1}{1 + x^2} - \frac{3}{2}$ arc $\tan x$. **21.** $\frac{1}{3}\{2$ arc $\tan\frac{x}{2} -$ arc $\tan x\}$.

$\left[\frac{x^2}{(x^2 + 1)(x^2 + 4)} = \frac{A}{(x^2 + 1)} + \frac{B}{x^2 + 4}\right.$; find A and B and integrate these fractions.] **22.** $\sin mx \sin nx = \frac{1}{2}\cos(m - n)x - \frac{1}{2}\cos(m + n)x$. Hence if $m \neq n$, $\int \sin mx \sin nx\, dx = \frac{1}{2}\int \cos(m - n)x\, dx - \frac{1}{2}\int \cos(m + n)x\, dx$. If $m = n$, $\int \sin mx \sin nx\, dx = \int \sin^2 mx\, dx = \frac{1}{2}\int (1 - \cos 2mx)\, dx$. The result follows directly from these integrals when taken between the limits given in the question. **23.** Express the given product as a sum or difference of cosines of multiple angles. **24.** This result follows from Example 9 above. **25.** If $\phi(x) = \phi(a + x)$ for *all* values of x, $\phi(x)$ must be a periodic function of period a, whence the result is obvious graphically. **26.** $\phi(a) \leqq \phi(x) \leqq \phi(b)$, whence the result. $\frac{d}{dx}\phi(x)$ must be positive for all values of x between (and including) a and b. **27.** $\phi(a)(b - a) > \int_a^b \phi(x)dx > \phi(b)(b - a)$. $\frac{d}{dx}\phi(x)$ must be negative when $a \leqq x \leqq b$. **28.** If $\phi(x)$ is negative between a and b and steadily increases, then $| \phi(a)(b - a) | > | \int_a^b \phi(x)\,dx | > | \phi(b)(b - a) |$. If $\phi(x)$ is negative between a and b and steadily decreases, then $| \phi(a)(b - a) | < | \int_a^b \phi(x)\,dx | < | \phi(b)(b - a) |$. When $\phi(x)$ changes sign between a and b, the integral may be positive, zero, or negative. If $\phi(x)$ decreases steadily from a to b, $\phi(a)(\xi - a) > \int_a^b \phi(x)\,dx$, where ξ is the point where $\phi(x)$ cuts the Ox axis. If $\phi(x)$ increases steadily from a to b, $\phi(a)(\xi - a) < \int_a^b \phi(x)\,dx$. **29.** Let $\psi(x)$ be a function continuous between a and b. Join the points $\{a, \psi(a)\}$ and $\{b, \psi(b)\}$ by a straight line. The gradient of this straight line is $\frac{\psi(b) - \psi(a)}{b - a}$. It is evident from a figure that there must be at least one point between a and b where the gradient of $\psi(x)$ is the same as the mean gradient of the straight line, hence if this point is the point $x = \xi$, $\psi'(\xi) = \frac{\psi(b) - \psi(a)}{(b - a)}$.

$$\therefore\ \psi(b) - \psi(a) = (b - a)\psi'(\xi). \quad \ldots\ldots\ldots\ldots(1)$$

Now $\psi(x)$ is the integral of $\psi'(x)$, $\therefore\ \psi(b) - \psi(a) = \int_a^b \psi'(x)\,dx$. $\therefore\ \int_a^b \psi'(x)\,dx = \psi'(\xi)\{b - a\}$. Putting $\psi'(x) = \phi(x)$ and $\xi = a + \theta(b - a)$ where θ is a number between 0 and 1, we get

$$\int_a^b \phi(x)\,dx = (b - a)\phi\{a + \theta(b - a)\}. \quad \ldots\ldots\ldots\ldots(2)$$

Equation (1) is also important. If we put $b = a + h$ we get

$$\psi(a + h) = \psi(a) + \psi'(\xi)\,h = \psi(a) + \psi'\{a + \theta h\}h. \ldots(3)$$

This is a very useful form of the theorem. **30.** $\sin^2\theta$ is always positive.

$$\therefore \sqrt{1 - \tfrac{1}{9}\sin^2\theta} \text{ lies between } 1 \text{ and } \sqrt{\frac{8}{9}}.$$

$$\therefore \int_0^{\frac{\pi}{2}} d\theta > \int_0^{\frac{\pi}{2}} \sqrt{1 - \tfrac{1}{9}\sin^2\theta}\, d\theta > \int_0^{\frac{\pi}{2}} \sqrt{\frac{8}{9}}\, d\theta.$$

$$\therefore \frac{\pi}{2} > \int_0^{\frac{\pi}{2}} \sqrt{1 - \tfrac{1}{9}\sin^2\theta}\, d\theta > \sqrt{\frac{8}{9}}\,\frac{\pi}{2},$$

i.e. $1\cdot571 > I > 1\cdot485$. If we take the mean, $1\cdot528$, the error cannot exceed $\frac{4\cdot3}{1\cdot528} = 2\cdot8\,\%$.

Exercise 9

1. $\frac{2}{3\sqrt{c}}[\overline{l + c}^{\frac{3}{2}} - c^{\frac{3}{2}}]$ where $\sqrt{ca} = \frac{4}{9}$. **2.** The integral required is $I = \int_0^l \sqrt{1 + Ax^{\frac{1}{2}}}dx$ where $A = \sqrt{a} \times \frac{25}{16}$. Put $1 + Ax^{\frac{1}{2}} = z^2$, when the indefinite integral becomes $\frac{4}{A^2}\int(z^4 - z^2)dz$, and $I = \frac{4}{A^2}\left[\frac{(1 + Ax^{\frac{1}{2}})^{\frac{5}{2}}}{5} - \frac{(1 + Ax^{\frac{1}{2}})^{\frac{3}{2}}}{3} - \frac{2}{15}\right]$ where $A = \sqrt{a} \times \frac{25}{16}$. **3.** The curves intersect above the x axis where $x = 0$ and $x = a$, hence the area required is $\int_0^a (\sqrt{2ax - x^2} - \sqrt{ax})dx$. Put $x = a(1 - \cos\theta)$ in the first integral and we get $a^2\left[\frac{\pi}{4} - \frac{2}{3}\right]$ for the area required. **4.** The graph is like a figure 8 on its side, ∞, the x axis passing through the centres of the two loops, and the y axis through the point where the loops meet. The area required is therefore given by $4\int_0^a \sqrt{\frac{a^2 - x^2}{a^2 + x^2}}\,x\,dx$ when a is 10. Put $x^2 = z$ and rationalize the numerator, and the indefinite integral is $\frac{1}{2}a^2 \arcsin\frac{x^2}{a^2} + \frac{1}{2}\sqrt{a^4 - x^4}$, whence the required area is $100(\pi - 2)$, when a is 10. **5.** The integral required is $2a\int_b^a \frac{\sqrt{2ax - x^2}}{x}dx$. Rationalize the numerator and we get as the indefinite integral $\sqrt{2ax - x^2} + a\int\frac{dx}{\sqrt{2ax - x^2}}$. The latter integral is $\arccos\left(1 - \frac{x}{a}\right)$, hence $\int\frac{\sqrt{2ax - x^2}}{x}\,dx$

$= \sqrt{2ax - x^2} + a \text{ arc } \cos\left(1 - \frac{x}{a}\right)$, and the required definite integral is $2a\left[\sqrt{2ax - x^2} + a \text{ arc } \cos\left(1 - \frac{x}{a}\right)\right]_b^a$. **6.** Integrate the expression for y between x_0 and x_1 to find the area accurately. Compare the result with those obtained by the rule, taking first the simple case when $n = 2$. **7.** $\pi r l$. **8.** $4\pi r^2$. **9.** $\frac{1}{3}\pi r^2 h$. **10.** $\frac{1}{3}\pi h^2(3r - h)$. **11.** $\frac{1}{2}\pi r^2 h$. **12.** $\frac{4}{3}\pi b^2 a$. **13.** $\frac{4}{3}\pi a^2 b$. **14.** $2\pi r a$.

Exercise 10

2. $6a \cos ax - 6a^2 x \sin ax - x^2 a^3 \cos ax$. **3.** $n^3 \sin mx \sin nx - 3mn^2 \cos mx \cos nx + 3m^2 n \sin mx \sin nx - m^3 \cos mx \cos nx$, i.e. $(n^3 + 3m^2 n) \sin mx \sin nx - (m^3 + 3mn^2) \cos mx \cos nx$. **5.** Differentiate y twice with respect to x, and add m^2 times y. The result will be found to be zero, as stated. **9.** When $f(x) = x^5$ every derivative is zero when $x = 0$ except the fifth, which is $5!$ whence the series reduces to x^5 as it should. **10.** $f(a + y) = \phi(y)$ since a is constant, and $\phi(y) = \phi(0) + \phi'(0)y + \frac{\phi''(0)y^2}{2!} + \ldots + \frac{\phi^{(n)}(0)y^n}{n!}$, i.e. $f(a + y) = f(a) + f'(a)y + \frac{f''(a)}{2!}y^2 + \ldots + \frac{f^{(n)}(a)y^n}{n!}$. **12.** Put $x + x^2 = z$ and expand by Ex. 11 above. Result is $1 + 4x + 10x^2 + 16x^3 + 19x^4 + 16x^5 + 10x^6 + 4x^7 + x^8$.

Exercise 11

4. The result follows from the working on pp. 208, 209, putting $-b^2 = b_1^2$. At the end of the major axis $(a, 0)$, $\rho = \frac{b^2}{a}$. **6.** (1) Differentiate y and x with respect to θ and divide the derivatives. The result is the value of dy/dx, i.e. $\tan\psi$. The result reduces at once to the value given by expressing it in terms of $\frac{\theta}{2}$. (2) Use the formula $s = \int_0^{x_0} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$ expressing $\frac{dy}{dx}$ and the integral in terms of θ. (3) $\delta s = \rho\,\delta\psi$, i.e. $4a \cos\psi\, d\psi = \rho\,\delta\psi$, whence the result. The geometrical result follows since $\rho = 2 \times 2a \cos\psi$. **7.** The tangential mass-acceleration along the direction of s increasing is $m\frac{d^2 s}{dt^2}$. The tangential force along the same direction is $-mg \sin\psi$.

$\therefore \frac{d^2s}{dt^2} + g \sin\psi = 0$ is the equation which determines the motion, by Newton's Second Law. But $\sin\psi = \frac{s}{4a}$ for a cycloid (Ex. 6 above), $\therefore \frac{d^2s}{dt^2} + \frac{g}{4a}s = 0$. If the body starts from rest (i.e. has zero velocity when $t = 0$) at $s = A$, it *might* move in accordance with $s = A \cos\omega t$, because this equation gives $s = A$ when $t = 0$ and $ds/dt = 0$ when $t = 0$. By differentiating this equation, it will be found to satisfy the equation of motion, if $\omega^2 = \frac{g}{4a}$ irrespective of the value of A. The radius of curvature at the point where the tangent makes angle ψ with OX is $4a \cos\psi$, whence the normal mass-acceleration is $\frac{mv^2}{4a \cos\psi}$ where v is the velocity of the particle at ψ. The reaction is therefore $\left(\frac{mv^2}{4a \cos\psi} + mg \cos\psi\right)$ allowing for the gravity reaction too. **8.** In $M = \frac{EI}{R}$; M, E, I are all constant, hence R is constant, and the centre line deflects into a circle. $y = \frac{1}{2}\frac{M}{EI}(x - l)x$, if $y = 0$ at both ends of the beam. If the weight is not negligible, we have to add (algebraically) the deflection for a uniformly distributed load (p. 214) to the deflection arising from the couples. **9.** $y = \frac{W}{2EI}\left(\frac{lx^2}{4} - \frac{x^3}{6}\right) - \frac{1}{48}\frac{Wl^3}{EI}$; $y_{\text{middle}} = -\frac{Wl^3}{48EI}$. **10.** $EIy = -\frac{W(l - x)^3}{6} - \frac{Wl^2x}{2} + \frac{Wl^3}{6}$; $y_{\text{end}} = -\frac{Wl^3}{3}$.

Exercise 12

1. (i) A minimum when $x = 3$, a maximum when $x = 1$ by applying the analytical test. (ii) $dy/dx = 0$ when $x = 0, 0, +3, -3$. The zero values of x make d^2y/dx^2 zero, and these are not turning values. Maximum value when $x = -3$; minimum value when $x = +3$. **2.** If R is the radius of the sphere and r of the base of the cone, h being the height of the cone, $r = h \tan\theta = R \sin 2\theta$. $\therefore$ vol. of cone $= \frac{\pi R^3}{3}\frac{\sin^3 2\theta}{\tan\theta}$.

The function of θ is the only variable part of this formula, and $\frac{\sin^3 2\theta}{8\tan\theta} = x^2 - 2x^4 + x^6$ where $x = \sin\theta$, whence we find that this is a maximum when $x^2 = \frac{1}{3}$, i.e. $\frac{\sin^3 2\theta}{\tan\theta} = \frac{32}{27}$, and the volume of the cone is $\frac{32}{81}\pi R^3$ and the radius ($r = R\sin 2\theta$) is $\frac{2\sqrt{2}}{3}R$.
3. A square should be cut out of each corner, the side of the square being $\frac{a + b - \sqrt{a^2 - ab + b^2}}{6}$, where a and b are the length and breadth of the piece of paper. **4.** Differentiating we get $\phi'(\theta) = \cos\theta - 2\sin 2\theta$. This expression equals zero when $\cos\theta = 4\sin\theta\cos\theta$, i.e. when $\cos\theta = 0$ or $\sin\theta = \frac{1}{4}$, i.e. when $\theta = 2\pi n \pm \frac{\pi}{2}$ (1) or $\theta = n\pi + (-1)^n\theta_1$ (2), where θ_1 is the angle lying between 0 and $\frac{\pi}{2}$ whose sine is $\frac{1}{4}$. On plotting the graph of $\sin\theta + \cos 2\theta$, it is easily seen that the values of θ given by (1) and (2) interlace. The first maximum occurs at θ_1, after which the turning values are a minimum and a maximum alternately. **5.** The corners of the triangle folded over and the corner of the unfolded page lie on a circle, whence the point of folding at the lower edge should be at the point $\frac{3}{4}$ along the lower edge. **6.** Express the distance between P and Q at time t and differentiate it to find the turning value. This method is long and clumsy. The following alternative method is briefer. Superimpose on the two bodies the velocity of one of the bodies A reversed. Then the body A is brought to rest, and the other body B moves with its own velocity, compounded with the velocity added, along the diagonal of the velocity parallelogram. The velocity relative to A (at rest) is then $\sqrt{u^2 + v^2 - 2uv\cos\theta}$. Now apply the principle of moments to the resultant velocity and the component velocities of B. **7.** If r is the horizontal range, the given equation shows that $r = 2h\sin 2\theta$, whence $dr/d\theta$ is zero when $\cos 2\theta = 0$, i.e. $\theta = 45^\circ$ for a turning value. That $\theta = 45^\circ$ gives a maximum value of r is obvious from the physics of the problem. **8.** The potential energy due to the charge of electricity is $\frac{ae^2}{r} = w_1$ and $\frac{dw_1}{dr} = -\frac{ae^2}{r^2}$, hence the equation for equilibrium becomes $\delta w = 2T\delta s - p\delta v - \frac{ae^2}{r^2}\delta r$. $\therefore 0 = 2T \times 8\pi r\delta r - p$

$\times\, 4\pi r^2\delta r - \frac{ae^2}{r^2}\delta r$, i.e. $p = \frac{4T}{r} - \frac{ae^2}{4\pi r^4}$, i.e. the equilibrium pressure is reduced by $\frac{ae^2}{4\pi r^4}$. **9.** 28″ vacuum. **10.** $\frac{dv}{dt} = \omega^2 r$ $(\cos\omega t + \frac{r}{l}\cos 2\omega t)$, whence turning values of v occur when $\cos\omega t + \frac{r}{l}\cos 2\omega t = 0$. Express $\cos 2\omega t$ in terms of $\cos^2\omega t$ and we get $\cos\omega t = \frac{l}{4r}\left[-1 \pm \sqrt{1 + \frac{8r^2}{l^2}}\right]$. The negative sign to the root is inadmissible as it would make $|\cos\omega t| >$ unity, in general. Hence $\omega t = \text{arc}\cos\left[\frac{l}{4r}\left(-1 + \sqrt{1 + \frac{8r^2}{l^2}}\right)\right]$, and maximum velocities occur when $t = \frac{1}{\omega}\left[2n\pi + \text{arc}\cos\frac{l}{4r}\left(-1 + \sqrt{1 + \frac{8r^2}{l^2}}\right)\right]$; 7·88 f.p.s.; angle turned through is 85°.

Exercise 13

1. (*a*) $\frac{1}{\sqrt{1 + x^2}}$; (*b*) $\frac{1}{\sqrt{x^2 + a^2}}$; (*c*) $\frac{n(\log_e x)^{n-1}}{x}$; (*d*) $\frac{2}{\sin 2x}$; (*e*) $-\sec x$; (*f*) $\operatorname{cosec} x$; (*g*) $e^x(\cos x - \sin x)$; (*h*) $e^{\sin x}(\cos^2 x - \sin x)$; (*i*) $\cot 2x$. **2.** $\int\frac{dx}{\sqrt{x^2 + a^2}} = \log_e\{x + \sqrt{x^2 + a^2}\}$; $\int\sec x\, dx = \log_e\left(\sqrt{\frac{1 - \sin x}{1 + \sin x}}\right) = \log_e \tan\left(\frac{\pi}{4} - \frac{x}{2}\right)$; $\int\operatorname{cosec} x\, dx = -\log_e\sqrt{\frac{1 - \cos x}{1 + \cos x}} = -\log_e \tan\frac{x}{2}$. **3.** Differentiate the right-hand side of the equation. **4** and **5.** By integration by parts, and the integrals obtained in questions (3) and (4). **6.** Integrate by parts. **7.** The integrand equals $\frac{3}{(x + 3)^2} + \frac{2}{(x + 3)} - \frac{2}{(x + 2)}$, hence the integral is $-\frac{3}{(x + 3)} + 2\log_e\left(\frac{x + 3}{x + 2}\right)$. **8.** $-\frac{5x + 12}{x^2 + 6x + 8} + \log_e\left(\frac{x + 4}{x + 2}\right)^2$. **9.** $\frac{8x^5}{1 + 2x^2} = 4x^3 - 2x + \frac{2x}{(1 + 2x^2)}$, whence the integral is $\frac{x^4}{8} - \frac{x^2}{8} + \frac{1}{8}\log_e\sqrt{1 + 2x^2}$. **10.** $\theta = A\cos\sqrt{\frac{g}{l}}t + B\sin\sqrt{\frac{g}{l}}t$. Prove by substitution. When $t = 0$, $\theta = \Theta$ and $\frac{d\theta}{dt} = 0$. $\frac{d\theta}{dt} = -A\sqrt{\frac{g}{l}}\sin\sqrt{\frac{g}{l}}t$

$+ B\sqrt{\frac{g}{l}} \cos\sqrt{\frac{g}{l}}t$. $\therefore$ o $= -A\sqrt{\frac{g}{l}} \sin 0^\circ + B\sqrt{\frac{g}{l}} \cos 0^\circ$. $\therefore$ B must be zero and $\Theta = A \cos 0^\circ$. $\therefore A = \Theta$, i.e. $\theta = \Theta \cos\sqrt{\frac{g}{l}}t$ is the solution required. The period of this function is 2π. If then τ is the periodic time, $\sqrt{\frac{g}{l}}\tau = 2\pi$. $\therefore \tau = 2\pi\sqrt{\frac{l}{g}}$.

11. Show that the force due to buoyancy is a restoring force and proportional to the depth of immersion, approximately. $\tau = 2\pi\sqrt{\frac{M}{ma^2g}}$, where m is the density of water. **12.** (1) If $v = \kappa s$, $v = 0$ when $s = 0$, and how can the body begin to move at all? (2) If $v = \kappa s$, then $\log_e s = \kappa t$, i.e. $\log_e\left(\frac{s_1}{s_2}\right) = \kappa(t_1 - t_2)$, i.e. the logarithm of the ratio of distances travelled should be proportional to the difference in times of transit. This is not true for motion due to gravity. **13.** Taking one of the supported ends as origin, the undeflected axis as axis of x, and the perpendicular through O in the plane of bending as the axis of y, the equation of the deflected axis is

$$\left(x - \frac{l}{2}\right)^2 = \frac{l^2}{4\Delta}(\Delta - y), \qquad \text{(1)}$$

where Δ is the deflection at the centre, initially. The total kinetic energy when the tube is crossing the x axis transversely is

$$\tfrac{1}{2}\rho\omega^2\int_0^l y^2\,dx, \qquad \text{(2)}$$

where ρ is the mass per foot, ω, the "angular velocity" of the harmonic motion, i.e. $2\pi n$ where n is the frequency, and y the initial co-ordinate of the deflected centre line of the tube. Integrating (2) by means of (1) and equating the result to $\mu\Delta^2$, the initial *potential* energy where μ is a constant, we get

$$\tau = 3 \cdot 25\sqrt{\frac{l\rho}{\mu}}, \qquad \text{(3)}$$

where τ is the required period. We can get μ approximately thus: When the tube is bent by a central concentrated load W, the potential energy is $\frac{1}{2}W\Delta g$ ft.-poundals, where g is the acceleration due to gravity. But in this case, the central deflection Δ is given by $\Delta = \frac{Wl^3}{48EI}$. $\therefore W = \frac{48EI\Delta}{l^3}$ and

$\frac{1}{2}\,\frac{48\text{EI}\Delta}{l^3}\,\Delta g = \mu\,\Delta^2.\quad \therefore\ \mu = \frac{24\,\text{EI}g}{l^3}.\quad \therefore\ \tau = 0{\cdot}664 l^2\sqrt{\frac{\rho}{\text{EI}g}}.$
In the example given $l = 10$ ft., $\rho = 0{\cdot}52$ lb. per foot, $\text{EI}g = 24000$, whence $\tau = 0{\cdot}309$ sec. It is a good practice to check "dimensions" of a formula of this kind $[\text{T}] = [\text{O}]\,[\text{L}^2]\,[\text{M L}^{-1}]^{\frac{1}{2}}\,[\text{M L}^{-2}\text{L}^4\,\text{L T}^{-2}]^{-\frac{1}{2}}$, i.e. $[\text{T}] = [\text{T}]$. The more accurate formula is $\tau = 0{\cdot}636 l^2\sqrt{\frac{\rho}{\text{EI}g}}$ when the correct deflection curve for a distributed load is used. (See Morley, *Strength of Materials*, p. 397, 1908 edition.) The effect of water in the tube is to increase the mass per unit length, without increasing the flexional rigidity. The period is therefore lengthened.

Exercise 14

1. The question leads at once to $\text{I}\frac{d\omega}{dt} = -\mu\omega$ (see p. 267), $\omega = \Omega e^{-\frac{\mu}{\text{I}}t}$, $\text{I} = 2240 \times 9$, $\omega/\Omega = 0{\cdot}5$, whence $\mu = 46{\cdot}5$ poundal ft. per radian per second. **2.** $e = \text{T}\frac{d\phi}{dt}$, p. 278. $\therefore\ ri = \text{T}\frac{d\phi}{dt}$. $\therefore\ d\phi = \frac{r}{\text{T}}i\,dt$. $\therefore\ \phi_2 - \phi_1 = \frac{r}{\text{T}}(\text{Q}_2 - \text{Q}_1)$. **3.** $\text{Q}_2 - \text{Q}_1 = \mu' a\sqrt{\frac{a}{\beta}}$, p. 295. $\therefore\ \phi_2 - \phi_1 = \mu'' a\sqrt{\frac{a}{\beta}}$ where $\mu'' = \frac{\mu' r}{\text{T}}$.
4. $\phi_2 - \phi_1 = \left(100 \times \frac{\pi \times 7{\cdot}5^2}{4}\right)$ lines. $\therefore\ \mu'' = \frac{100 \times \pi \times 7{\cdot}5^2}{4 \times 65} = 68$. Double. **5.** $\phi = \frac{100 \times \pi \times 4^2}{4} = 400\pi$. $\therefore\ \frac{d\phi}{dt} = 4000\pi$. $\therefore\ e = 100 \times 4000\pi \times 10^{-8}$ volts $\left[e = \text{T}\frac{d\phi}{dt} \times 10^{-8}\right.$ volts$\left.\right] = 4\pi \times 10^{-3}$ volts. $\therefore\ i = \frac{e}{r} = \frac{4\pi \times 10^{-3}}{50} = 25 \times 10^{-5}$ ampere. **6.** Clearly the flux through the coil varies sinusoidally, hence, in the usual notation, $\phi = \Phi\cos\omega t$. $\therefore\ \frac{d\phi}{dt} = -\Phi\omega\sin\omega t$. $\therefore\ e_{\text{max.}} = \Phi\omega\text{T} \times 10^{-8}$ volts numerically. $\Phi = \left[0{\cdot}18 \times \frac{\pi \times 30^2}{4}\right]$, $\text{T} = 20$, $\omega = 2\pi\frac{2000}{60}$. $\therefore\ e_{\text{max.}} = 5{\cdot}3 \times 10^{-3}$ volt. **7.** Consider a uniform magnetic field of strength H, and a perfect conductor carrying a current i and of length l, lying in the field perpendicular to it and moving with velocity v perpendicular to its length l and to the direction of the field.

Clearly $\frac{d\phi}{dt} = Hlv$. $\therefore$ ei, the rate of working, is $Hlvi \left[e = T\frac{d\phi}{dt}\right]$. But this quantity is Flv, where F is the resultant transverse force on the conductor per centimetre. $\therefore$ $Flv = Hilv$, i.e. $F = Hi$. Hence, in a uniform field, the force on a conductor is proportional to the field strength and the current. A ballistic galvanometer is so constructed that the coil moves in a uniform radial field. Hence the force on a vertical element of the coil is Hli, and the total torque is therefore $\Sigma Hlir$, where r is the radius of the coil. By symmetry every element produces the same torque, $\therefore$ $\Sigma Hlir = HLir$, where L is the total length of the elements, i.e. the electromagnetic torque is $h'i$, where $h' = HLr$, a constant; hence the torque applied to the needle is not now zero but $h'i$, and we get, dividing h' by the moment of inertia of the needle, $\frac{d^2\theta}{dt^2} + \omega^2\theta = hi$, where $h = \frac{h'}{I}$, I being the moment of inertia of the needle. The complete solution of the equation is $\theta = A \sin\omega t + B \cos\omega t + \frac{hi}{\omega^2}$, if i is constant. Whence $\theta = 0$ when $t = 0$ gives $B = -\frac{hi}{\omega^2}$, and $\frac{d\theta}{dt} = 0$ when $t = 0$ gives $A = 0$. $\therefore$ $\theta = \frac{hi}{\omega^2}[1 - \cos\omega t]$.

Exercise 15

1. If a is original concentration, and x the amount transferred in time t, then $\frac{1}{t}\log_e\left(\frac{a}{a-x}\right) = \kappa$ if the action be monomolecular (p. 308). Using this formula, we get the table

t intervals		$a - x$		κ
1	......	88	......	(0·128)
2	......	72	......	0·168
4	......	53	......	0·159
7	......	29·5	......	0·174
8	......	25·2	......	0·172
11	......	15·2	......	0·171
13	......	11·1	......	0·169

which shows that κ is practically constant. (Lewis, *A System of Physical Chemistry*, Vol. I, p. 433.) 2. If reaction is monomolecular, we have $\frac{1}{t}\log_e\left(\frac{a}{a-x}\right) = \kappa$. Testing the numbers, we find κ is nearly constant at 15×10^{-4}. (Lewis, *A System* *of Physical Chemistry*, Vol. I, p. 436.) The chemical equation merely gives the proportions in which the cane sugar changes into dextrose and laevulose. **3.** The formula is $\frac{1}{t}\log_e\left(\frac{C_0}{C_t}\right) = \kappa_\theta$. If $C_t = \frac{1}{2}C_0$, $\frac{C_0}{C_t} = 2$. $\therefore t = \frac{\log_e 2}{\kappa\theta}$, whence $t_1 = 7{\cdot}1$ wk. and $t_2 = 21{\cdot}8$ min. **4.** If the formula is monomolecular, the formula is

$$\kappa_1 = \frac{1}{t}\log_e\frac{a}{a-x}, \quad\ldots\ldots\ldots\ldots(1)$$

but if it is quadrimolecular,

$$\kappa_4 = \frac{1}{t}\,\frac{1}{3}\left\{\frac{1}{(a-x)^3} - \frac{1}{a^3}\right\},\ \text{see p. 309.}\ \ldots\ldots\ldots(2)$$

The numbers work out as follows:

t (hr.)		$\kappa_1 \times 10^3$		$\kappa_4 \times 10^2$
0	......	—	......	—
7·83	......	2·36	......	1·73
24·17	......	2·37	......	2·01
41·25	......	2·35	......	2·29
63·17	......	2·38	......	2·88
89·67	......	2·41	......	3·85

κ_1 is more nearly constant than κ_4 is. The equations may be

$$1\ PH_3 \rightarrow P + 3\ H.$$
$$\left.\begin{array}{l} 4\ P \rightarrow P_4 \\ 2\ H \rightarrow H_2 \end{array}\right\}\text{extremely rapidly.}$$

In this case, the first (monomolecular) reaction would dominate the rate of the complete reaction. (Lewis, *A System of Physical Chemistry*, Vol. I, p. 443.) **5.** (*a*) Differentiate the given expression and substitute. The two sides of the given equation will be found to be identical. (*b*) When $x = 0$, $\sin nx = 0$ and $c = c_0$; when $x = l$, $\sin nl = 0$. $\therefore nl = m\pi$ ($m = 0, 1, 2, 3 \ldots$). $\therefore c = c_0 + ae^{\frac{-Dm^2\pi^2}{l^2}t}\sin\left(\frac{m\pi}{l}x\right)$. (*c*) Each term separately satisfies the given differential equation, which is of the first degree.

Exercise 16

1. 0·57 ($\gamma = 1{\cdot}4$). **2.** 0·52 ($\gamma = 1{\cdot}4$). **3.** 0·32. The available heat energy is 371 B.Th.U., and the heat absorbed is 1158 B.Th.U. **4.** The temperature of the mixture is

$5{\cdot}73^\circ$ C. The ice gains $\left(\frac{L}{\theta} + \log_e \frac{\theta_2}{\theta}\right)$, i.e. $0{\cdot}313$ unit of entropy. The water loses $20 \log_e \frac{283}{278{\cdot}7} = 0{\cdot}308$ unit of entropy. Net gain of whole system $= 0{\cdot}005$ unit of entropy. **5.** Consider the gas A. At any moment it is slowly expanding against a force p, due to the piston. This expansion is reversible, for if p is changed to $p + \delta p$, the expansion would cease and compression would begin. The expansion is also adiabatic; it is therefore isentropic. Similarly for gas B. The friction of the piston balances the difference in the gas pressures, and work has been done irreversibly against friction and converted into heat. Hence the entropy of the piston must have increased. Hence the total entropy of the system has increased. **6.** Work area is $[p_1 v_1 + \int_{v_1}^{v_2} p\,dv - p_2 v_2]$ where $pv^n = \kappa$. Whence work done is $\frac{n}{n-1} p_2 v_2 \left[\left(\frac{p_1}{p_2}\right)^{\frac{n-1}{n}} - 1\right]$ ft.-lb., i.e. work done per cubic foot ($v_2 = 1$) is $144 \frac{n}{n-1} p_2 \left[\left(\frac{p_1}{p_2}\right)^{\frac{n-1}{n}} - 1\right]$ if p_2 is in pounds per *square inch*. **7.** Work per minute $= 144 \frac{n}{n-1} V p_2 \left[\left(\frac{p_1}{p_2}\right)^{\frac{n-1}{n}} - 1\right]$. If $p_1 = p_2 + \delta p_2$, $\left[\left(\frac{p_1}{p_2}\right)^{\frac{n-1}{n}} - 1\right] = \frac{n-1}{n} \frac{\delta p_2}{p_2}$ nearly, whence H.P. $= \frac{144}{33000} V p_2$. **8.** Follows at once from Ex. 6. **9.** $v_2 = 1{\cdot}2$ c. ft.; $v_1 = 0{\cdot}2$ c. ft.; $r = 6$; $\theta_2 = 600^\circ$ F. abs.; $p_2 = 14{\cdot}7$ lb./sq. in.; gas used is $0{\cdot}06$ c. ft. per cycle. Cal. Val. $= 600$ B.Th.U. per cubic foot $= 36$ B.Th.U. per cycle. 1 c. ft. of air at 32° F. and $14{\cdot}7$ lb. per square inch weighs $0{\cdot}0807$ lb. Hence: (i) Weight of gas in cylinder is $\cdot 0807 \times \frac{493}{600} \times 1{\cdot}2 = 0{\cdot}0793$ lb. (ii) During compression $pv^{1{\cdot}38} =$ a constant. $\therefore P_{\text{compression}} = 14{\cdot}7 \times 6^{1{\cdot}38} = 174{\cdot}2$ lb./sq. in., and, since $\frac{PV}{T} =$ constant, $\theta_{\text{compression}} = 600 \times \frac{173}{14{\cdot}7} \times \frac{\cdot 2}{1{\cdot}2} = 1185^\circ$ F. abs. (iii) The rise of temperature $\Delta\theta$, due to combustion, is given by $0{\cdot}0793 \times 0{\cdot}18 \times \Delta\theta = 36$. $\therefore \Delta\theta = 2523^\circ$ F. $\therefore \theta_{\text{expansion}} = 2523 + 1185 = 3708^\circ$ F. abs. The corresponding pressure is: (iv) $P_{\text{expansion}} = 174{\cdot}2 \times \frac{3708}{1185} = 543$ lb./sq. in. (v) The pressure at release ($pv^n = \kappa$) is given by $P_{\text{release}} = \frac{543}{6^{1{\cdot}38}} = 46$ lb./sq.

in. The corresponding temperature $\left(\frac{pv}{\theta} = \text{constant}\right)$ is given by $\theta_{\text{release}} = 3708 \times \frac{46}{543} \times \frac{1\cdot 2}{\cdot 2} = 1878°$ F. abs. (vi) The efficiency is $\{1 - (\frac{1}{6})^{0\cdot 38}\} = 49\cdot 4\%$. (vii) The Carnot efficiency is $\frac{3708 - 600}{3708} = 83\cdot 8\%$. **10.** 0·75, 0·52; internal energy at A, 89·1; at B, 66·4 B.Th.U.; work done, 2040 ft.-lb.; heat lost to walls, 20·1 B.Th.U. **11.** 6·3 h.p. **12.** Volume of the reservoir is 9·05 c. ft. 9·05 c. ft. of air at 500 lb./sq. in. and 275° C. abs. weigh 24·7 lb.; hence 24·7 lb. of air remain in the reservoir. This air occupies 2·8 c. ft. at the initial state. If this air expanded adiabatically, the final temperature would be $\theta_2 = 288 \times 3\cdot 23^{-0\cdot 38} = 184°$ C. abs., for 3·23 is the ratio of the change of volume. The actual temperature is 275° C. abs., hence heat from sea is $24\cdot 7 \times 0\cdot 18 \times (275 - 184) = 404$ C.H.U., i.e. 404 pound-calories.

MISCELLANEOUS

1. (i) Put $\frac{1}{x(x-1)^2} = \frac{A}{(x-1)^2} + \frac{B}{x-1} + \frac{C}{x}$. To find A, multiply both sides by $(x-1)^2$, and then put $x = 1$; thus $A = 1$. To find C, multiply by x and put $x = 0$; thus $C = 1$. Then B is easily found to be -1. Integral is $-\frac{1}{x-1} + \log x - \log(x-1)$. (ii) Integrate by parts. Integral is $\frac{1}{4}x^4 \log x - \frac{1}{16}x^4$. (iii) Put $x - \frac{3}{2} = z$. Integral is $\log(x - \frac{3}{2} + \sqrt{x^2 - 3x + 2})$. (iv) Put $\tan\frac{x}{2} = z$; indefinite integral becomes $\int \frac{2dz}{9 + z^2}$. **2.** (i) Put $e^x = z$. Integral is arc $\tan(e^x)$. (ii) Integral $= \int \frac{1-x}{\sqrt{(1-x^2)}}\,dx = \int \frac{dx}{\sqrt{(1-x^2)}} - \int \frac{xdx}{\sqrt{(1-x^2)}}$ $=$ arc $\sin x + \sqrt{(1-x^2)}$. (iii) $\int \left(\frac{2}{x-2} - \frac{1}{x^2+4}\right) dx$ $= 2\log(x-2) - \frac{1}{2}$ arc $\tan\frac{1}{2}x$. **3.** (i) Derivative is $\frac{1}{\sqrt{(1-x^2)}}$. (ii) The function is arc $\sin x$. (iii) Integrate $(1-x^2)^{-\frac{1}{2}}$ once, result is arc $\sin x + C$; integrate this again, by parts, function required is x arc $\sin x + \sqrt{(1 - x^2)} + Cx + D$. **4.** (i) Express integrand in form $\frac{A}{x^3} + \frac{B}{x^2} + \frac{C}{x} + \frac{D}{x - 1}$. Integral $= \frac{1}{2}\frac{1}{x^2} + \frac{1}{x} - \log x + \log(x - 1)$. (ii) $\frac{1}{2}e^x(\sin x - \cos x)$. (iii) From repeated integration by parts, integral $= e^x(x^3 - 3x^2 + 6x - 6)$. **5.** $\frac{dy}{dx} = 1 - \frac{a^{\frac{1}{2}}}{x^{\frac{1}{2}}} = -\frac{y^{\frac{1}{2}}}{x^{\frac{1}{2}}}$. Equation of tangent at (h, k) is $\frac{x}{h^{\frac{1}{2}}} + \frac{y}{k^{\frac{1}{2}}} = a^{\frac{1}{2}}$. $OP + OQ = a^{\frac{1}{2}}h^{\frac{1}{2}} + a^{\frac{1}{2}}k^{\frac{1}{2}} = a$. **6.** $\int e^{ax}\sin^n x\,dx$ $= \frac{1}{a}e^{ax}\sin^n x - \int \frac{1}{a}e^{ax}n\sin^{n-1}x\cos x\,dx$, by parts, $= \frac{1}{a}e^{ax}\sin^n x - \frac{n}{a^2}e^{ax}\sin^{n-1}x\cos x + \frac{n}{a^2}\int e^{ax}\{-\sin^n x + (n - 1)\sin^{n-2}x\cos^2 x\}dx$. The integrated terms vanish for $x = 0$ and $x = \beta$. In last integral write $\sin^{n-2}x\cos^2 x = \sin^{n-2}x(1 - \sin^2 x) = \sin^{n-2}x - \sin^n x$. Thus $\int_0^\beta e^{ax}\sin^n x\,dx = \frac{n}{a^2}\int_0^\beta e^{ax}\{(n - 1)\sin^{n-2}x - n\sin^n x\}\,dx$. Hence $\left(1 + \frac{n^2}{a^2}\right)\int_0^\beta e^{ax}\sin^n x\,dx = \frac{n(n - 1)}{a^2}\int_0^\beta e^{ax}\sin^{n-2}x\,dx$. **7.** (i) $\int \sin^m x\cos^n x\,dx = \int \sin^{m-1}x\frac{d}{dx}(-)\frac{\cos^{n+1}x}{n + 1}\,dx = -\frac{1}{n + 1}\sin^{m-1}x\cos^{n+1}x + \frac{1}{n + 1}\int \cos^{n+1}x\,(m - 1)\sin^{m-2}x\cos x\,dx$. In last integral write $\cos^{n+2}x = \cos^n x - \cos^n x\sin^2 x$, and proceed as in Ex. 6. (ii) By (i), $\int_0^{\frac{\pi}{2}}\cos^4 x\sin^2 x\,dx = \frac{1}{6}\int_0^{\frac{\pi}{2}}\cos^4 x\,dx$ $= \frac{1}{6}\,\frac{3\cdot 1}{4\cdot 2}\,\frac{\pi}{2}$ (p. 161) $= \frac{\pi}{32}$. (iii) $\int_0^{\frac{\pi}{2}}\cos^5 x(1 - \cos^2 x)\frac{d}{dx}(-\cos x)\,dx$ $= \int_0^1 u^5(1 - u^2)\,du = \frac{1}{6} - \frac{1}{8} = \frac{1}{24}$. **8.** (1) For any value of x, $f(x)$ is increasing with x if $f'(x)$ is $+$, decreasing with x if $f'(x)$ is $-$. (2) Where the graph of $f'(x)$ crosses the axis of x, $f(x)$ has a turning value, a maximum or a minimum according as the graph of $f'(x)$ crosses from above or below as x increases. (3) Turning points on graph of $f'(x)$ correspond to points of inflexion on the graph of $f(x)$. In a cone, $V = \frac{1}{3}\pi r^2 h$, $S = \pi r\sqrt{r^2 + h^2}$. $\therefore S^2 = \pi^2 r^4 + 9V^2 r^{-2}$, which is a maximum when $4\pi^2 r^3 - 18V^2 r^{-3} = 0$, which gives $h = \sqrt{2}r$. Then angle of sector, in radians, $= \frac{\text{length of base}}{\text{slant side}} = \frac{2\pi r}{\sqrt{(r^2 + h^2)}} = \frac{2\pi}{\sqrt{3}}$.

9. (i) Put $x - \frac{1}{2}(a+b) = u$, so that $(x-a)(x-b) = u^2 - \frac{1}{4}(a-b)^2$. (ii) Take 10 ordinates, viz. at $x = 1, 1{\cdot}9, 2{\cdot}8$, &c. The rule then gives $2{\cdot}312$; the correct value is $2{\cdot}303$. **10.** (i) $\int\left(\frac{2}{(x-1)^2} - \frac{1}{x^2}\right)dx = \frac{1}{x} - \frac{2}{x-1}$. (ii) $\int\frac{dx}{\sqrt{\{(x-1)^2-1\}}} = \log(x - 1 + \sqrt{x^2 - 2x})$. (iii) Put $\tan x = u$. Integral $= \frac{1}{ab}\operatorname{arc}\tan\left(\frac{a\tan x}{b}\right)$. (iv) Put $x = a\sin\theta$, then $\tan\theta = t$. The indefinite integral $= \int\frac{a^2\cos^2\theta\, d\theta}{c^2 - a^2\sin^2\theta} = \int\left\{\frac{1}{1+t^2} - \frac{c^2-a^2}{(c^2-a^2)t^2+c^2}\right\}dt = \operatorname{arc}\tan t - \frac{\sqrt{c^2-a^2}}{c}\operatorname{arc}\tan\frac{t\sqrt{c^2-a^2}}{c}$. **11.** (i) $\int_0^1 f(x)\,dx = \text{A} + \frac{1}{2}\text{B} + \frac{1}{3}\text{C}, f(0) = \text{A}, f(\frac{1}{2}) = \text{A} + \frac{1}{2}\text{B} + \frac{1}{4}\text{C}$, $f(1) = \text{A} + \text{B} + \text{C}$, and result follows at once. (ii) Put $x = \frac{1}{2} + u$, and express $\phi(x)$ in powers of u. **12.** (i) $\frac{1}{\sin x\cos x}$ or $\frac{2}{\sin 2x}$. (ii) $e^{x^2+1}\frac{2x^3}{(x^2+1)^2}$. (iii) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$, and either test result by substitution, or else eliminate a, b from the equations giving $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$. **13.** (i) $\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = \frac{\sin\theta}{1+\cos\theta} = \frac{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}{2\cos^2\frac{1}{2}\theta} = \tan\frac{1}{2}\theta$. (ii) $\left(\frac{ds}{d\theta}\right)^2 = \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = a^2\{(1+\cos\theta)^2 + \sin^2\theta\} = 2a^2(1+\cos\theta) = 4a^2\cos^2\frac{\theta}{2}$. Hence $\frac{ds}{d\theta} = 2a\cos\frac{\theta}{2}$, $s = 4a\sin\frac{\theta}{2} = 4a\sin\phi$. (iii) $\rho = \frac{ds}{d\phi} = 4a\cos\phi$, intercept on normal $= \frac{2a - y}{\cos\phi} = \frac{a(1+\cos\theta)}{\cos\phi} = \frac{2a\cos^2\phi}{\cos\phi} = 2a\cos\phi$.
14. $\int\frac{x + \frac{1}{2}(\beta+\gamma) + \{\alpha - \frac{1}{2}(\beta+\gamma)\}}{\sqrt{[\{x+\frac{1}{2}(\beta+\gamma)\}^2 - \frac{1}{4}(\beta-\gamma)^2]}}dx = \{(x+\beta)(x+\gamma)\}^{\frac{1}{2}} + \{\alpha - \frac{1}{2}(\beta+\gamma)\}\log[x + \frac{1}{2}(\beta+\gamma) + \{(x+\beta)(x+\gamma)\}^{\frac{1}{2}}]$. **15.** (i) $x\log x - x$. (ii) $\frac{1}{2}x^2\log x - \frac{1}{4}x^2$. (iii) $\frac{1}{a}xe^{ax} - \frac{1}{a^2}e^{ax}$. (iv) From the graph of $\log\text{X}$ it can be seen at once that $\log\text{X} < \text{X} - 1$; for the line $\text{Y} = \text{X} - 1$ is the tangent to the graph at $(1, 0)$, and the log graph lies below this tangent. Put $1 + \frac{1}{2}x$ for X; then $\log(1+\frac{1}{2}x) < \frac{1}{2}x$ and $x\log(1+\frac{1}{2}x) < \frac{1}{2}x^2$, from which the result stated follows. **16.** It is easier here to deal with

$\frac{1}{y} = x + 1 + \frac{1}{x} = z$ say, and find the minimum and maximum values of z. For $x = -1$, $y = -1$, a minimum; and for $x = 1$, $y = \frac{1}{3}$, a maximum. **17.** Deal with the reciprocal of $\cos\psi$, proving $(1-\sigma)^2 \sec^2\psi = \frac{\sin^2\theta + \sigma^2\cos^2\theta}{\sin^2\theta\cos^2\theta} = \frac{1}{\cos^2\theta} + \frac{\sigma^2}{\sin^2\theta}$ $= 1 + \tan^2\theta + \sigma^2(1 + \cot^2\theta)$. Put $\tan^2\theta = x$, then $1 + x + \sigma^2\left(1 + \frac{1}{x}\right)$ is to be a minimum. **18.** (i) Put integrand $= \frac{A}{(x+1)^2} + \frac{B}{x+1} + \frac{C}{x-1}$ (see Ex. 1 (i)); integral $= -\frac{1}{2}\frac{1}{x+1}$ $+ \frac{1}{4}\log\frac{x-1}{x+1}$. (ii) $\int \frac{1}{2}(1 + \cos 2x)\,dx = \frac{1}{2}x + \frac{1}{4}\sin 2x$. (iii) First formula is $n\int \frac{\sin(2n+1)x - \sin(2n-1)x}{\sin x}\,dx = \sin 2nx$, which is proved at once from the trigonometrical identity $\sin(2n+1)x - \sin(2n-1)x = 2\cos 2nx \sin x$. For second formula use the identity $\sin^2(n+1)x - \sin^2 nx = \sin(2n+1)x \sin x$. For the definite integrals, first formula gives $s_{n+1} - s_n = 0$, so that $s_n = s_1$, or $s_n = \frac{\pi}{2}$. The second formula then gives $v_n - v_{n-1} = \frac{\pi}{2}$. In this result put $n = 1, 2, 3$, &c., and add. **19.** $\pi\int_0^1 y^2 dx$ $= \frac{\pi}{24}$. **20.** Put $(p, p_1, p_2)^{\frac{n-1}{n}}$ respectively $= q, q_1, q_2$. Then $\frac{q_1}{q} + \frac{q}{q_2} - 2$ is to be a minimum. **21.** If r is the radius and v the velocity at time t, then the law that change of momentum = applied force gives $\frac{d}{dt}(r^3 v) = r^3 g$. Also $r = r_0 + kt$, where $k = \frac{dr}{dt}$, a constant. Hence $\frac{d}{dr}(r^3 v) = \frac{1}{k}\frac{d}{dt}(r^3 v)$, or $r^3 v$ $= \frac{g}{k}\int_{r_0}^{r} r^3 dr = \frac{g}{4k}(r^4 - r_0^4)$. If $v = \frac{dx}{dt} = k\frac{dx}{dr}$, this gives x $= \frac{g}{4k^2}\int_{r_0}^{r}\left(r - \frac{r_0^4}{r^3}\right)dr = \frac{g}{8k^2}\left(r^2 + \frac{r_0^4}{r^2} - 2r_0^2\right)$. This equation will give r_1, the value of r when the drop reaches the ground, where $x = h$. The quadratic for r_1 is $(r_1^2 - r_0^2)^2 = \frac{8k^2hr_1^2}{g}$, so that $r_1^2 - r_0^2 = r_1\sqrt{\frac{8k^2h}{g}}$ and $r_1 = k\sqrt{\frac{2h}{g}} + \sqrt{\left(\frac{2k^2h}{g} + r_0^2\right)}$.

Using foot-second units, we have $k = \frac{1}{12}\cdot 10^{-4}$, $h = 6400$, $g = 32$, $r_0 = \frac{1}{12} \times 0\cdot04$. Thus the final rad. $= 0\cdot04205$ in., and then from the equation $r = r_0 + kt$, we find the time $= 20\cdot5$ sec. **22.** (i) Equation of tangent at (a, b) is $y - b = -3a^2(x - a)$. (ii) $a = 0\cdot58$, $b = 7\cdot8$, intercepts $= 8\cdot4$. (iii) Area $= 12$. **23.** $M\ddot{y} = -Mr\omega^2(\cos\omega t + \frac{r}{l}\cos 2\omega t)$. This is numerically greatest when the cosines are both equal to 1, i.e. when $t = 0$, $\frac{2\pi}{\omega}$, $\frac{4\pi}{\omega}$, &c. The maximum value of the inertia force is $\therefore$ $\frac{M\omega^2 r}{g}\left(1 + \frac{r}{l}\right)$ lb., if M is in pounds. **24.** (i) $\frac{100r}{l} \div \left(1 + \frac{r}{l}\right)$, or $\frac{100r}{l}$ nearly. (ii) For two reasons: (*a*) the velocity coefficient is $\frac{r}{2l}$ and the inertia coefficient $\frac{r}{l}$; (*b*) $\sin 2\omega t$ is very small when the velocity is a maximum, whereas $\cos 2\omega t$ is equal to 1 when the acceleration is a maximum. **25.** $7\cdot56$ ft.-tons or 16,900 ft.-lb. **26.** (i) See p. 292. The T.P. occur where $\frac{d\theta}{dt} = 0$ or $-k\sin nt + n\cos nt = 0$. Hence $\tan nt = \frac{n}{k}$. If we take an acute angle α so that $\frac{n}{k} = \tan\alpha$, then $nt = \alpha$, $\alpha + \pi$, $\alpha + 2\pi$, $\alpha + 3\pi$, &c., give the T.P. The maxima and minima occur alternately, θ being $+$ at the maxima, and $-$ at the minima. Thus maxima occur when $t = \frac{\alpha}{n}$, $\frac{\alpha}{n} + \frac{2\pi}{n}$, $\frac{\alpha}{n} + \frac{4\pi}{n}$, . . . The values of θ, or $\theta_0 e^{-kt}\sin nt$, for these values of t are in G.P. with common ratio $e^{-k\cdot 2\pi/n}$. (ii) Given $\frac{2\pi}{n} = 1$, and $(e^{-k\cdot 2\pi/n})^4 = \frac{1}{4}$; if T is time required in seconds, $\tan nT = \frac{n}{k}$. But $e^{4k} = 4$, $k = \frac{1}{4}\log_e 4 = \cdot3466$, $\tan 2\pi T$ $= \frac{2\pi}{\cdot3466} = 18\cdot13$, $2\pi T = \frac{\pi}{2} - \text{arc}\tan\frac{1}{18\cdot13}$, $T = \frac{1}{4} - \frac{1}{18\cdot13 \times 2\pi}$ nearly $= 0\cdot241$. **27.** Energy consumed is proportional to the resistance R, and annual charges for copper are proportional to the cross-section, i.e. inversely proportional to R. Hence annual charges $= a + \frac{b}{R}$, value of energy lost $= cR$, where

a, b, c are constants. $\frac{b}{R} + cR$ is a minimum when $-\frac{b}{R^2} + c = 0$ or $\frac{b}{R} = cR$. Let section $= x$ sq. cm. Interest on cost of copper $= \frac{x \times 9 \times 2{\cdot}2046 \times 6 \times 240}{1000 \times 2240}$ pence, cost of energy consumed $= \frac{1 \times 1{\cdot}6 \times 10^{-6} \times 500 \times 500 \times 12 \times 365 \times 3}{x \times 1000 \times 4}$ pence. Equating, we get $x^2 = 103$ and $x = 10{\cdot}15$. $\therefore$ section $= 10{\cdot}15$ sq. cm. **28.** Take moments about centre of bowl. **29.** $m\frac{dv}{dt} = F - av^2$. At full speed v_0, $\frac{dv}{dt} = 0$. $\therefore$ $F = av_0^2$, $m\frac{dv}{dt} = a(v_0^2 - v^2)$, $\frac{a}{m}\frac{dt}{dv} = \frac{1}{v_0^2 - v^2}$, $\frac{a}{m}t = \frac{1}{2v_0}\log_e\frac{v_0 + v}{v_0 - v}$. But $Fv_0 = P$, $\therefore$ $P = av_0^3$. Hence $t = \frac{m}{2av_0}\log_e\frac{v_0+v}{v_0-v} = \frac{mv_0^2}{2P}\log_e\frac{v_0+v}{v_0-v}$. In example, $t = \frac{2000 \times 66 \times 66}{2 \times 15 \times 550 \times 32}\log_e\frac{45+30}{45-30} = \frac{33}{2}\log_e 5 = \frac{33}{2} \times 1{\cdot}61 = 26{\cdot}6$ sec. **30.** See pp. 292–5, and p. 296, Exs. 2, 3. When a current of i amperes is reversed, the formula for change of flux is $\frac{2i}{10} \times 4\pi \times$ turns in primary $\times$ turns in secondary $\times \frac{\text{area of 1 turn of secondary}}{\text{length of primary}} = 1{\cdot}6 \times \pi \times 500 \times 500 \times \pi \times (15/4)^2 \div 40 = 1{,}390{,}000$ lines. We have now to find the number of divisions, undamped, corresponding to 65 divisions, damped. Now each maximum swing $=$ preceding maximum swing $\times e^{-2\lambda}$. $\therefore$ 10th maximum swing : 1st $= e^{-18\lambda}$. Hence $\frac{65}{12} = e^{18\lambda}$. But 1st undamped swing : 1st damped swing $= e^{\frac{\lambda}{2}} = (\frac{65}{12})^{\frac{1}{36}}$. $\therefore$ 1st undamped swing $= 68{\cdot}1$. $\therefore$ Flux per undamped scale division $= \frac{1{,}390{,}000}{68{\cdot}1} = 20{,}400$ lines. If resistance were halved, first fling would be doubled, $136{\cdot}2$ divisions. **31.** (i) See p. 151. (ii) If $pv^c = A$, work done $= \frac{A}{c-1}(v_1^{1-c} - v_2^{1-c}) = \frac{1}{c-1}(p_1v_1^cv_1^{1-c} - p_2v_2^cv_2^{1-c}) = \frac{1}{c-1}(p_1v_1 - p_2v_2)$. (iii) $p_2 = p_1\left(\frac{v_1}{v_2}\right)^c = \frac{15}{2^{1{\cdot}41}} = 5{\cdot}64$ lb. per sq. in. Work done, by (ii), $= 7{\cdot}54$ ft.-lb. **32.** (i) If $p + \delta p =$ pressure at eighth

$z + \delta z$, then $\delta p = -g\rho\delta z$ nearly, when δz is small, p being in absolute units. Hence $\frac{dp}{dz} = -g\rho = -gAp$, since ρ is proportional to p when temperature is constant. $\frac{dp}{p} = -gAdz$ gives $\log p + C = -gAz$; and if $p = p_0$ when $z = 0$, then $\log\frac{p}{p_0} = -gAz$ and $\frac{p}{p_0} = e^{-gAz}$. (ii) Upward thrust = weight of air displaced. This weight of air, when lowest point is at height h, is $\int_h^{h+d}\rho S dz$, where S = cross-section of sphere at height z, and d = diameter of sphere; also $\rho = \rho_0 e^{-\mu z}$, so that weight of air displaced $= \rho_0\int_h^{h+d} Se^{-\mu z}dz$. Put $z = h + \zeta$, and this becomes $\rho_0\int_0^d S' e^{-\mu h}e^{-\mu\zeta}d\zeta = \rho_0 e^{-\mu h} \times \int_0^d S' e^{-\mu\zeta}d\zeta = e^{-\mu h}$ × weight of air displaced when sphere rests on ground $= e^{-\mu h}(M + m)g$. (S′ is the cross-section at height ζ above lowest point of sphere.) Sphere will rest at height h if this last expression = weight of sphere = Mg. Hence $e^{-\mu h} = \frac{M}{M+m}$. $\therefore \mu h = \log_e\frac{M+m}{M}$. **33.** If a drop of water is introduced into an enclosure containing vapour and water in equilibrium, evaporation will diminish the surface of the drop and therefore its surface energy, but will increase the remaining part of the energy of liquid and vapour. Equilibrium will be reached when the total potential energy is a minimum. Now consider 1 lb. of a vapour obeying Boyle's Law. If we reckon its potential energy from a standard density ρ_0, the increase in potential energy when it expands to density ρ is given by $W = -\int_{v_0}^{v} p\,dv = -R\theta\int_{v_0}^{v}\frac{dv}{v} = -R\theta\log_e\frac{v}{v_0} = R\theta\log_e\frac{\rho}{\rho_0}$; since $v\rho = v_0\rho_0 = 1$. Consider 1 lb. of water. We suppose it to be incompressible, so that its density σ is constant, and it has no elastic potential energy. We may therefore write $W_{\text{vapour}} = R\theta\log_e\frac{\rho}{\rho_0} + f(\theta)$ per lb.; $W_{\text{water}} = \phi(\theta)$ per lb. Suppose there are ξ lb. of vapour and η lb. of water, and in the first place let the water be free from surface tension and electrification. Then the whole potential energy W is given by

$$W = \xi R\theta\log_e\frac{\rho}{\rho_0} + \xi f(\theta) + \eta\phi(\theta). \qquad \text{.............(1)}$$

Suppose everything takes place at constant temperature. When there is equilibrium we must have $\frac{dW}{d\xi} = 0$. Now, if M is the total mass, and V the total volume, supposed constant, then $\xi + \eta = M$ and $\frac{\xi}{\rho} + \frac{\eta}{\sigma} = V$. Thus from (1), $\frac{dW}{d\xi} = R\theta \log_e \frac{\rho}{\rho_0} + \xi R\theta \frac{1}{\rho}\frac{d\rho}{d\xi} + f(\theta) - \phi(\theta)$, for $\eta = M - \xi$ and $\frac{d\eta}{d\xi} = -1$. Now $\frac{\xi}{\rho} = V - \frac{\eta}{\sigma}$, $\frac{d}{d\xi}\left(\frac{\xi}{\rho}\right) = \frac{1}{\sigma}$, and $\therefore \frac{\xi}{\rho}\frac{d\rho}{d\xi} = 1 - \frac{\rho}{\sigma}$. Thus, from (1),

$$\frac{dW}{d\xi} = R\theta \log_e \frac{\rho}{\rho_0} - R\theta\frac{\rho}{\sigma} + \psi(\theta). \quad \text{...............(2)}$$

Thus, in equilibrium,

$$R\theta \log_e \frac{\rho}{\rho_0} - R\theta\frac{\rho}{\sigma} + \psi(\theta) = 0. \quad \text{...............(3)}$$

This is the normal equation of equilibrium. But if some physical cause, such as surface tension or electrification, makes the potential energy of the system increase by X, then we must add on the left of (3) a term $\frac{dX}{d\xi}$, so that

$$R\theta \log_e \frac{\rho'}{\rho_0} - R\theta\frac{\rho'}{\sigma} + \psi(\theta) + \frac{dX}{d\xi} = 0, \quad \text{...........(4)}$$

where ρ' is the new equilibrium value of ρ; we may put $\rho' = \rho + \delta\rho$. From (3) and (4), by subtraction, we get $R\theta$ $(\log_e\rho - \log_e\rho') + R\theta\frac{\rho' - \rho}{\sigma} - \frac{dX}{d\xi} = 0$, or $\left(\text{since } \delta \log\rho = \frac{1}{\rho}\delta\rho\right)$ $-R\theta\frac{\delta\rho}{\rho} + R\theta\frac{\delta\rho}{\sigma} = \frac{dX}{d\xi}$, or $R\theta\frac{\sigma - \rho}{\sigma\rho}\delta\rho = -\frac{dX}{d\xi}$. Now $p = R\theta\rho$. $\therefore \delta p = R\theta\,\delta\rho$, and we get finally

$$\delta p = -\frac{\sigma\rho}{\sigma - \rho}\frac{dX}{d\xi}. \quad \text{......................(5)}$$

Surface Tension.—$X = 4\pi r^2 T$, $\xi = M - \frac{4\pi}{3}r^3\sigma$. $\therefore \frac{dX}{d\xi} = \frac{8\pi r T dr}{-4\pi r^2 \sigma dr} = -\frac{2T}{r\sigma}$. Hence $\delta p = \frac{2\rho}{\sigma - \rho}\frac{T}{r}$. *Electrification.*—$X = \frac{e^2}{2\kappa r}$, $\frac{dX}{d\xi} = -\frac{e^2}{2\kappa r^2}dr \div -4\pi r^2\sigma dr = \frac{e^2}{8\pi\kappa r^4\sigma}$, $\delta p = -\frac{\rho}{\sigma - \rho}\frac{e^2}{8\pi\kappa r^4}$. The effect of electrification is therefore to diminish the equilibrium vapour density, and so increase the tendency to the deposition of vapour on the drop. An electrified drop of rain

should therefore be larger than an unelectrified one. This, perhaps, explains the large size of the drops of rain which fall in thunderstorms. See Poynting and Thomson's *Physics* (Properties of Matter), p. 166, Seventh Edition; also J. J. Thomson's *Applications of Dynamics to Physics and Chemistry*, 1888, p. 158, &c. **34.** (i) See p. 349. (ii) 0·614. (iii) $1 - \left(\frac{1}{r}\right)^{\gamma - 1} = 0{\cdot}652$.
35. (i) The dryness fraction at the final point must be such that the entropy in the initial and final states are identical. If q is the dryness fraction, then $(1 - q)$ lb. is water, q lb. steam. Entropy $= \int_{\theta_0}^{\theta} \frac{\sigma d\theta}{\theta} + \frac{qL}{\theta}$, assuming that we have 1 lb. of water at θ_0; for we may pass from the initial to the final state by first raising the water to θ without evaporation, and then evaporating q lb. of it at θ. Thus $\phi = \sigma \log_e \frac{\theta}{\theta_0} + \frac{qL}{\theta}$. Suppose the condition of the mixture is given at θ_1 and we wish the value of q at θ_2. Since ϕ is constant, we have $\sigma \log_e \theta_2 + \frac{q_2 L_2}{\theta_2} = \sigma \log_e \theta_1 + \frac{q_1 L_1}{\theta_1}$, which gives q_2. (ii) If ϕ and q are constant, $\sigma \log_e \theta - \sigma \log_e \theta_0 + \frac{qL}{\theta} = a''$, a constant. $\therefore \sigma \log_e \theta = -\frac{qL}{\theta} + a'$, $\log_e \theta = -\frac{qL}{\sigma\theta} + \log_e a$, $\theta = ae^{-\frac{qL}{\sigma\theta}}$. **36.** (i) See p. 340. (ii) $dQ = C_v d\theta + p dv = C_v d\theta + \frac{\kappa dv}{v^n}$. $\therefore \int dQ = C_v(\theta_2 - \theta_1) + \frac{1}{n-1}(p_1 v_1 - p_2 v_2)$ (see Ex. 31 above) $= C_v(\theta_2 - \theta_1) + \frac{1}{n-1}(R\theta_1 - R\theta_2) = \left(C_v - \frac{R}{n-1}\right)(\theta_2 - \theta_1)$.
37. Consider the cycle for 1 lb. of water: (1) Heat it as water from T to T + δT. (2) Expand it to vapour at $p + \delta p$. (3) Cool it as vapour from T + δT to T. (4) Condense it to water at T. Then work done per cycle $= [\{p + \left(\frac{\partial p}{\partial T}\right)_v \delta T\} - p](V - \omega)$ or $(V - \omega)\left(\frac{\partial p}{\partial T}\right)_v \delta T$. By Carnot's Theorem (p. 359) this $= \frac{\delta T}{T} \times$ heat taken in $= \frac{\delta T}{T} \times L$, L being in work units. Divide by δT, noting that $\left(\frac{\partial p}{\partial T}\right)_v$ and $\left(\frac{\partial T}{\partial p}\right)_v$ are reciprocals, and the result follows. **38.** Apply the result of Ex. 37 to 1 gramme-molecule (p. 301). Let v_2 = volume of vapour and

neglect the volume of the liquid. Then, taking θ for temperature, $L = v_2\theta\frac{dp}{d\theta}$, $v_2 = \frac{R\theta}{p}$, so that $L = \frac{R\theta^2}{p}\frac{dp}{d\theta}$. If P is the internal work of evaporization, $L = P + pv$, $\therefore P + R\theta = \frac{R\theta^2}{p}\frac{dp}{d\theta}$, $\left(\frac{P}{\theta^2} + \frac{R}{\theta}\right)d\theta = \frac{R}{p}dp$, or by integration, $-\frac{P}{\theta} + R\log\theta = R\log p + C$. $\therefore R\log\frac{p_1}{\theta_1} - R\log\frac{p_2}{\theta_2} = -\frac{P}{\theta_1} + \frac{P}{\theta_2}$. $\therefore R\log\frac{p_2\theta_1}{p_1\theta_2} = P\left(\frac{1}{\theta_1} - \frac{1}{\theta_2}\right)$. Now, by the Kinetic Theory of Gases, p is proportional to $N\sqrt{\theta}$, where N is the number of molecules leaving the surface per square centimetre per second. Thus $R\log\frac{N_2\sqrt{\theta_2}\theta_1}{N_1\sqrt{\theta_1}\theta_2} = P\left(\frac{1}{\theta_1} - \frac{1}{\theta_2}\right)$. In the electric case, if e = charge on an electron, then $i = Ne$ and $R = 2$. Hence $2\log\frac{i_2\sqrt{\theta_1}}{i_1\sqrt{\theta_2}} = P\left(\frac{1}{\theta_1} - \frac{1}{\theta_2}\right)$. Hence $\log\frac{\sqrt{\theta_1}}{i_1} - \log\frac{\sqrt{\theta_2}}{i_2} = \frac{1}{2}P\left(\frac{1}{\theta_1} - \frac{1}{\theta_2}\right)$ or $\log\frac{\sqrt{\theta_1}}{i_1} - \frac{1}{2}\frac{P}{\theta_1} = \log\frac{\sqrt{\theta_2}}{i_2} - \frac{1}{2}\frac{P}{\theta_2}$, and $\therefore$ each is constant. $\therefore \frac{\sqrt{\theta}}{i} = ke^{\frac{P}{2\theta}}$ or $i = D\theta^{\frac{1}{2}}e^{-\frac{P}{2\theta}}$ (*Electrical Properties of Flames and Vapours*, H. A. Wilson). **39.** The vehicle of energy is the unit quantity of electricity. Consider a cycle: (1) Allow unit quantity of electricity to pass through the cell under electric force $E + \left(\frac{\partial E}{\partial T}\right)\delta T$, at constant temperature $T + \delta T$. (2) Allow the temperature of the cell to fall to T. (3) Pass unit quantity back through cell under force E. (4) Bring cell back to initial temperature $T + \delta T$. The electrical work done $= E + \left(\frac{\partial E}{\partial T}\right)\delta T - E$ or $\frac{\partial E}{\partial T}\delta T$ ergs, and this $= \frac{\delta T}{T} \times$ (heat taken in) by Carnot's Theorem (compare Ex. 37) or (heat taken in) $= T\frac{\partial E}{\partial T}$. Now some of this work is derived internally by chemical action, so that net input of heat $= E + \frac{\partial E}{\partial T}\delta T - \lambda$, or $E - \lambda$ approximately. Hence $E - \lambda = T\frac{\partial E}{\partial T}$. **40.** Put a film through the following cycle: (1) Stretch it at temperature $T + \delta T$ to area $A + \delta A$. (2) Cool it at $A + \delta A$ to T. (3) Contract it at T to A. (4) Warm it at A to $T + \delta T$. Surface tension in (1) is $S + \left(\frac{\partial S}{\partial T}\right)\delta T$. Work done *by* surface tension in (3) is $2S\delta A$ (considering both sides of film); in (1) it is $-2(S + \frac{\partial S}{\partial T}\delta T)\delta A$. $\therefore$ Net work done $= -2\frac{\partial S}{\partial T}\delta T\delta A$. If $H + \delta H$ is the total heat absorbed at $T + \delta T$ and H the heat rejected at T, then (p. 360) $\frac{H + \delta H}{T + \delta T} = \frac{H}{T}$. $\therefore \frac{\delta H}{\delta T} = \frac{H}{T}$, and δH = net work done $= -2\frac{\partial S}{\partial T}\delta T\delta A$. $\therefore H = T\frac{\delta H}{\delta T}$ or $H = -2T\frac{\partial S}{\partial T}\partial A$. In forming a film of area A we have therefore to supply energy (as work and heat) $= 2SA - 2T\frac{\partial S}{\partial T}A$. Hence surface energy per square centimetre (one side) $= S - T\frac{\partial S}{\partial T}$. **41.** Let specific heats in A, B be σ_1, σ_2; Peltier effect coefficients P at hot, P_0 at cold junction; temperature of hot junction T, of cold T_0. Suppose unit quantity of electricity to pass from the hot to the cold junction through A. Measure all energy in ergs. Then work done = E. Heat absorbed at junctions $= P - P_0$. Heat absorbed by A $= \int_T^{T_0}\sigma_1 dT = -\int_{T_0}^{T}\sigma_1 dT$. Heat absorbed by B $= \int_{T_0}^{T}\sigma_2 dT$. *First Law* of Thermodynamics gives $E = P - P_0 - \int_{T_0}^{T}(\sigma_1 - \sigma_2)dT$. *Second Law* $(\Sigma\frac{dQ}{T} = 0)$ gives $\frac{P}{T} - \frac{P_0}{T_0} - \int_{T_0}^{T}\frac{\sigma_1}{T}dT + \int_{T_0}^{T}\frac{\sigma_2}{T}dT = 0$. By differentiating these, we get (i) $\frac{dE}{dT} = \frac{dP}{dT} - (\sigma_1 - \sigma_2)$; (ii) $\frac{d}{dt}\left(\frac{P}{T}\right) = \frac{\sigma_1 - \sigma_2}{T}$. Hence, in (i), $\frac{dE}{dt} = \frac{dP}{dt} - T\frac{d}{dt}\left(\frac{P}{T}\right)$, or $\frac{dE}{dT} = \frac{P}{T}$, i.e. $P = T\frac{dE}{dT}$. Then, in (ii) $\sigma_2 - \sigma_1 = -T\frac{d}{dt}\frac{dE}{dt} = -T\frac{d^2E}{dT^2}$. **42.** If there are n positive ions and n negative ions per cubic centimetre, $\frac{dn}{dt} = -\theta n^2$. $\therefore -\frac{1}{n^2}\frac{dn}{dt} = \theta$, $\frac{1}{n} + c = \theta t$, $\frac{1}{n_2} - \frac{1}{n_1} = \theta t$. **43.** $\frac{dc}{dt} = -\kappa c^n$,

$\log\left(-\frac{dc}{dt}\right) = \log\kappa + n \log c$, $\therefore \log\left(-\frac{dc}{dt}\right)_v - \log\left(-\frac{dc}{dt}\right)_V$ $= n(\log c_v - \log c_V)$. In 1st experiment, mean concentration $= 171 \cdot 15$, mean velocity $= \frac{188 \cdot 84 - 153 \cdot 46}{23} = \frac{35 \cdot 38}{23}$. In 2nd, mean concentration $= 77 \cdot 52$, mean velocity $= \frac{2 \cdot 97}{20}$. Hence $n = \left(\log\frac{35 \cdot 38}{23} - \log\frac{2 \cdot 97}{20}\right) \div (\log 171 \cdot 15 - \log 77 \cdot 52) = 2 \cdot 95$. The mean concentration and mean velocity are approximately the concentration and velocity at the middle of the interval of time. See Van 't Hoff, *Lectures on Theoretical and Physical Chemistry* (1898), p. 200. **44.** (i) See Chapter XV. (ii) Equation may be written $\kappa\frac{dt}{dx} = \frac{1}{(C-A)(A-B)}\left(\frac{1}{B-x} - \frac{1}{A-x}\right)$ $+ \frac{1}{(C-B)(C-A)}\left(\frac{1}{C-x} - \frac{1}{B-x}\right)$. Integrate with respect to x from 0 to x. $\therefore \kappa t = \frac{1}{(C-A)(A-B)}\left\{\log\frac{A-x}{B-x} - \log\frac{A}{B}\right\}$ $+$ a similar term $= \frac{1}{(C-A)(A-B)}\left\{\log\frac{B(A-x)}{A(B-x)}\right\} + \ldots$, which is the result stated. **45.** (i) $\frac{dx}{dt} = \kappa(A-x)^3$. $\therefore \kappa t$ $= \int_0^x \frac{dx}{(A-x)^3} = \frac{1}{2}\left\{\frac{1}{(A-x)^2} - \frac{1}{A^2}\right\}$. (ii) Here $A = \frac{1}{10}$, so that, for a trimolecular reaction, $2\kappa t = \frac{100}{(1-10x)^2} - 100$. The values of κ calculated from the data and this equation are 1·04, 0·99, 1·03, 1·02, 1·06, 1·01. (iii) The chemical equation is quantitatively correct, but need not represent the actual reaction, which may be a combination of slow and swift exchanges. The result indicates that the dominant change is a slow trimolecular one, which is followed by swift adjustments to bring the final result into accord with the stoichiometrical equation (or equation of relative proportions) given. **46.** $0 \cdot 00303 = K_1 \times C_{acid} \times C_{alcohol} = K_1 \times 1 \times 12 \cdot 756$. $\therefore K_1 = \cdot 000238$ in mols per litre — minute units. **47.** $K_2 = \cdot 000996 \div (1 \times 12 \cdot 215)$ $= \cdot 0000815$. **48.** (i) $C_{eth.} = 0 \cdot 7144$, $C_{acid} = 1 - 0 \cdot 7144$ $= 0 \cdot 2856$, $C_{alcohol} = 12 \cdot 756 - 0 \cdot 7144 = 12 \cdot 0416$, C_{water} $= 12 \cdot 756 + 0 \cdot 7144 = 13 \cdot 4704$. (ii) In equilibrium, the reaction velocities in the two directions are equal, i.e. $K_1 \times C_{acid}$

$\times$ C$_{\text{alcohol}}$ = $K_2 \times$ C$_{\text{eth.}}$ $\times$ C$_{\text{water}}$. (iii) $\frac{0\cdot7144 \times 13\cdot4704}{0\cdot2856 \times 12\cdot0416} = 2\cdot80$, $\frac{K_1}{K_2}$ (Exs. 46, 47) $= \frac{\cdot000238}{\cdot0000815} = 2\cdot92$. **49.** (i) Plot $\log_{10}K$ against T. The four points will be found to lie nearly in a line. Draw a line to pass through the points as nearly as possible. Take two points some distance apart on this line, and find a and b by substituting the co-ordinates of these points in the equation $\log_{10}K = a + bT$. Various equally good results, differing slightly from each other, may be got. One result is $\log_{10}K = -5\cdot79 + 0\cdot043T$. Another method, not so accurate, is to determine a and b from two of the points given by the table, without using a graph. (ii) Rise of temperature enormously increases the speed of reactions. In the present case, e.g., if the speed is 3K at T′, we have $\log_{10}3k = -5\cdot79 + 0\cdot043T'$, and by subtraction $\log_{10}3k - \log_{10}k = 0\cdot043(T' - T)$, or $T' - T = \frac{\log_{10}3}{0\cdot043} = 11\cdot1$, so that a rise of 11° C. trebles the reaction velocity. A rule that applies to many cases is that the speed of reaction is doubled or trebled when the temperature is raised by 10° C. **50.** If x = concentration of inverted sugar at time t, a = initial concentration of cane sugar, then $\frac{dx}{dt} = \kappa(a - x)$ and $\kappa t = \log\frac{a}{a - x}$. When $x = \frac{9}{10}a$, $\kappa t = \log_e 10$.

The times are $\therefore \frac{2\cdot302}{\cdot765}$ and $\frac{2\cdot302}{35\cdot5}$ min.

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